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Introduction to Pluripotential Theory

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Abstract. Pluripotential Theory is the study of the "fine properties" of plurisubharmonic functions on domains in $\mathbb{C}^n$ as well as on complex manifolds. These functions appear naturally in Complex Analysis of Several Variables in connection with holomorphic functions. Indeed they appear as weights of metrics in the L^2-estimates of Hörmander for the solution to the Cauchy-Riemann equation on pseudoconvex domains culminating with the solution of the Levi problem (see [**Hor90**]).

They appear also in Kähler geometry as potentials for (singular) Kähler metrics on compact Kähler manifolds and as local weights for singular hermitian metrics on holomorphic line bundles. Here local plurisubharmonicity corresponds to (semi)-positivity of the curvature form of the corresponding singular metric.

Recently Pluripotential theory has found many interesting applications in Complex Algebraic Geometry ([**Dem13**]) as well as in Kähler geometry (e.g. the Calabi conjecture on Kähler singular varieties, the existence of singular Kähler-Einstein metrics, etc...). All these problems boil down to solving degenerate complex Monge-Ampère equations ([**GZ17**]).

The main goal of this course is to give an elementary introduction to this theory as developed by E. Bedford and B.A. Taylor in the late seventies and early eighties ([**BT76**], [**BT76**]). From their definition it follows that plurisubharmonic functions are subharmonic with respect to infinitely many Kähler metrics. Therefore the positive cone of plurisubharmonic functions can be viewed as an infinite intersection of "half spaces", hence it is of nonlinear nature. It turns out that the study of plurisubharmonic functions involves a fully nonlinear second order partial differential operator, called the complex Monge-Ampère operator, a nonlinear generalization of the Laplace operator from one complex variable.

We will first recall some elementary facts from logarithmic potential theory in the complex plane focusing on the Dirichlet problem for the Laplace operator. Then we will introduce the complex Monge-Ampère operator acting on bounded plurisubharmonic functions on domains in $\mathbb{C}^n$ and study its continuity properties. Finally we will we apply these results to solve the Dirichlet Problem for degenerate complex Monge-Ampre equations in strictly pseudo-convex domains in $\mathbb{C}^n$ using the Peron method which was introduced by Bremmermann in this context ([**Bre59**]).

The material of this course is taken essentially from [**GZ17**] (see also [**Klim91**]).

Contents

CHAPTER 1

Logarithmic Potential Theory

1. Introduction

The potential theory comes from mathematical physics, in particular, from electrostatic and gravitational problems and find applications in the probability theory, scattering theory, biological systems and many other branches of science. During the last century the classical potential theory and the non-linear potential theory has occupied an important place in mathematics. The term potential theory arises from the fact that, in 19^{th} century physics, the fundamental forces of nature were believed to be derived from potentials which satisfied Laplace's equation in $\mathbb{R}^3$. Hence, potential theory was the study of functions that could serve as potentials.

It turns out that these functions are subharmonic in the whole space $\mathbb{R}^3$ and concersly any subharmonic is (up to an harmonic function) equal to the Newton potential of its Riesz measure in any subdomain $D \Subset \mathbb{R}^3$. Therefore in Mathematics, Classical Potential Theory aims in studing the "fine properties" of subharmonic functions in domains of the euclidean space $\mathbb{R}^n$, using potentials associated to distributions of masses or charges.

Logarithmic Potential Theory is the study of harmonic and subharmonic functions in domains in $\mathbb{C} \simeq \mathbb{R}^2$. In this case there is a deep interplay with the theoryof holomorphic functions of one complex variable. Logarithmic Potential Theory turns out to have many important applications in Complex Analysis, Approximation Theory ([**ST97**]), Complex Dynamics and Functional Analysis ([**Ra95**].

2. Harmonic functions

2.1. Definitions and basic properties. Let $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ be a domain. Recall that a function $h : \Omega \to \mathbb{R}$ is *harmonic* if h is C^2-smooth and satisfies the Laplace equation

$$\Delta h = 0$$

in Ω, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}},$$

is the Laplace operator in $\mathbb{C} = \mathbb{R}^2$.

It follows from the Cauchy-Riemann equations that if $f:\Omega\longrightarrow\mathbb{C}$ is a holomorphic function then its real part $h=\Re e f$ is harmonic. The Cauchy formula shows that f has the mean-value property: for any closed disc $\bar{\mathbb{D}}(a,r)\subset\Omega$,

$$f(a)=\int_0^{2\pi} f(a+re^{i\theta})\frac{d\theta}{2\pi}. \tag{2.1}$$

Conversely any harmonic function is locally the real part of a holomorphic function, hence harmonic functions satisfy the mean-value property. The latter actually characterizes harmonic functions:

PROPOSITION 1.1. *Let $h:\Omega\longrightarrow\mathbb{R}$ be a continuous function in Ω. The following properties are equivalent:*

(i) the function h is harmonic in Ω;

(ii) for any $a\in\Omega$ and any disc $\mathbb{D}(a,r)\subset\Omega$ there is a holomorphic function f in the disc $\mathbb{D}(a,r)$ such that $h\equiv\Re e f$ in $\mathbb{D}(a,r)$;

(iii) the function h satisfies the mean-value property (2.1) at each point $a\in\Omega$ and for any $r>0$ such that $\bar{\mathbb{D}}(a,r)\subset\Omega$;

(iv) the function h satisfies the mean-value property (2.1) at each point $a\in\Omega$, for $r>0$ small enough.

In particular harmonic functions are real analytic hence C^∞-smooth.

PROOF. We first show the implication $(i)\Longrightarrow(ii)$. We need to prove that for a fixed disc $D=\mathbb{D}(a,r)\Subset\Omega$ there exists a smooth function g in D such that $h+ig$ is holomorphic in D. This boils down to solving the equation

$$dg=-\frac{\partial h}{\partial y}dx+\frac{\partial h}{\partial x}dy=:\alpha$$

in D. The 1-form α is closed in Ω since h is harmonic,

$$d\alpha=\Delta h\,dx\wedge dy\equiv 0.$$

The existence of g therefore follows from Poincaré lemma.

The implication $(ii)\Longrightarrow(iii)$ follows from the Cauchy formula as we have already observed, while the implication $(iii)\Longrightarrow(iv)$ is obvious.

It remains to show $(iv)\Longrightarrow(i)$. We first prove that h is actually smooth in Ω. Let $\rho:\mathbb{C}\longrightarrow\mathbb{R}^+$ be a radial function with compact support in the unit disc $\mathbb{D}$ such that $\int_{\mathbb{C}}\rho(z)\,d\lambda(z)=2\pi\int_0^1\rho(r)rdr=1$. We consider, for $\varepsilon>0$,

$$\rho_\varepsilon(z):=\varepsilon^{-2}\rho(z/\varepsilon)\quad\text{so that}\quad\int_{\mathbb{C}}\rho_\varepsilon(z)d\lambda(z)=1.$$

Set $h_\varepsilon:=h\star\rho_\varepsilon$ for $\varepsilon>0$ small enough. These functions are smooth and we claim that $h_\varepsilon=h$ in $\Omega_\varepsilon=\{z\in\Omega\,|\,\mathrm{dist}(z,\partial\Omega)>\varepsilon\}$ for $\varepsilon>0$

small enough. Indeed integrating in polar coordinates and using the mean value property for h we get

$$h_\varepsilon(a) = \int_0^1 r\rho(r)dr \int_0^{2\pi} h(a + \varepsilon re^{i\theta})d\theta = 2\pi h(a) \int_0^1 r\rho(r)dr = h(a).$$

Therefore $h = h_\varepsilon$ is smooth in Ω_ε.

Fix now $a \in \Omega$ and use Taylor expansion of h in a neighborhood of a: for $|z - a| = r << 1$

$$h(z) = h(a) + \Re P(z - a) + \frac{r^2}{2}\Delta h(a) + o(r^2),$$

where P is a quadratic polynomial in z such that $P(0) = 0$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})d\theta = h(a) + \frac{r^2}{2}\Delta h(a) + o(r^2),$$

hence

$$\Delta h(a) = \lim_{r\to 0^+} \frac{2}{r^2}\left(\int_0^{2\pi} h(a + re^{i\theta})\frac{d\theta}{2\pi} - h(a)\right) = 0,$$

by the mean value property. Thus h is harmonic in Ω. □

Let $\mathcal{D}(\Omega)$ denote the space of complex valued smooth functions with compact support in Ω and let $\mathcal{D}'(\Omega)$ denote the space of distributions (continuous linear forms on $\mathcal{D}(\Omega)$).

Recall that a function $f \in L^1_{loc}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$,

$$T(f) : \mathcal{D}(\Omega) \ni \chi \mapsto \int_\Omega \chi f d\lambda_2 \in \mathbb{C},$$

where $d\lambda_2$ denotes the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$.

This defines a distribution on Ω i.e. a continuous linear operator on the space $\mathcal{D}(\Omega)$ for the topology of local uniform convergence up to any order with uniform control on the supports. Observe that the mapping

$$T : L^1_{loc}(\Omega) \ni f \longmapsto T(f) \in \mathcal{D}'(\Omega)$$

is injective, so that we can identify f with $T(f)$ and consider f as a distribution acting on test functions by integration.

Actually this is a Radon measure on Ω i.e. a continuous linear operator on the space $\mathcal{K}(\Omega)$ of continuous test functions for the topology of local uniform convergence with uniform control on the supports.

Weyl's lemma shows that harmonic distributions are harmonic meaning that they are induced by harmonic functions:

Lemma 1.2. *(Weyl's lemma). Let $T \in \mathcal{D}'(\Omega)$ be a harmonic distribution on Ω. Then there is a unique harmonic function h in Ω such that $T = T(h)$ in $\mathcal{D}'(\Omega)$.*

This is a particular cas of a general property, called the (hypo)ellipticity of the Laplace operator.

Proof. Consider radial mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$ as above and set $T_\varepsilon := T \star \rho_\varepsilon$. Then T_ε is a smooth function in Ω_ε which satisfies $\Delta T_\varepsilon = (\Delta T) \star \rho_\varepsilon = 0$ in Ω_ε, hence it is a harmonic function in Ω_ε.

The proof of the previous proposition shows that for $\varepsilon, \eta > 0$,

$$T_\varepsilon = T_\varepsilon \star \rho_\eta = T_\eta \star \rho_\varepsilon = T_\eta$$

weakly in $\Omega_{\varepsilon+\eta}$. Letting $\varepsilon \to 0$ we obtain $T = T_\eta$ in the weak sense of distributions in Ω_η. Therefore as $\eta \to 0^+$ the functions T_η glue into a unique harmonic function h in Ω such that $T = T_h$ in Ω. □

2.2. Poisson formula and Harnack inequalities. The *Poisson formula* is a reproducing formula for harmonic functions:

Proposition 1.3. *(Poisson formula). Let $h : \bar{\mathbb{D}} \longrightarrow \mathbb{R}$ be a continuous function which is harmonic in $\mathbb{D}$. Then for all $z \in \mathbb{D}$*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta.$$

Proof. We reduce to the case when h is harmonic in a neighborhood of $\bar{\mathbb{D}}$ by considering $z \longmapsto h(rz)$ for $0 < r < 1$ and letting r increase to 1 in the end.

For $z = 0$ the formula above is the mean value property in the unit disc. Fix $a \in \mathbb{D}$ and let f_a be the automorphism of $\mathbb{D}$ sending 0 to a,

$$f_a(z) := \frac{z + a}{1 + \bar{a}z}.$$

The function $h \circ f_a$ is harmonic in a neighborhood of $\bar{\mathbb{D}}$, hence

$$h(a) = h \circ f_a(0) = \int_{\partial\mathbb{D}} h \circ f_a(z) d\sigma$$

The change of variables $\zeta = f_a(z)$ yields $z = f_{-a}(\zeta)$ and

$$h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta,$$

as desired. □

The following are called Harnack's inequalities:

Corollary 1.4. *Let $h : \bar{\mathbb{D}} \longrightarrow \mathbb{R}^+$ be a non-negative continuous function which is harmonic in $\mathbb{D}$. For all $0 < \rho < 1$ and $z \in \mathbb{D}$ with $|z| = \rho$, we have*

$$\frac{1 - \rho}{1 + \rho} h(0) \leq h(z) \leq \frac{1 + \rho}{1 - \rho} h(0).$$

Proof. Fix $z \in \mathbb{D}$ such that $|z| = \rho$ and observe that

$$\frac{1 - \rho}{1 + \rho} \leq \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \leq \frac{1 + \rho}{1 - \rho}.$$

Since $h \geq 0$ in $\partial\mathbb{D}$ we can multiply these inequalities by $h(e^{i\theta})$ and integrate over the unit circle. Poisson formula and the mean value property for h thus yield

$$\frac{1-\rho}{1+\rho}h(0) \leq h(z) \leq \frac{1+\rho}{1-\rho}h(0).$$

□

2.3. The maximum Principle. Harmonic functions satisfy the following fundamental maximum principle:

THEOREM 1.5. *Let $h : \Omega \to \mathbb{R}$ be a harmonic function.*
1. If h admits a local maximum at some point $a \in \Omega$ then h is constant in a neighborhood of a.
2. For any bounded subdomain $D \Subset \Omega$ we have

$$\max_{\bar{D}} h = \max_{\partial D} h.$$

Moreover $h(z) < \max_{\partial D} h$ for all $z \in D$ unless h is constant.

PROOF. 1. Assume there is a disc $\mathbb{D}(a,r) \subset \Omega$ s.t. $h(z) \leq h(a)$ for all $z \in \mathbb{D}(a,r)$. Fix $0 < s < r$ and note that $h(a) - h(a+se^{i\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$. The mean value property yields

$$\int_0^{2\pi} \left(h(a) - h(a+se^{i\theta})\right) d\theta = 0.$$

Since h is continuous we infer $h(a) - h(a+se^{i\theta}) = 0$ for all $\theta \in [0, 2\pi]$, hence h is constant as claimed.

2. By compactness there exists $a \in \bar{D}$ such that $h(a) = \max_{\bar{D}} h$. If $a \in D$ the previous case shows that h is constant in a neighborhood of a. Therefore the set $A := \{z \in D; h(z) = h(a) = \max_{\bar{D}} h\}$ is open, non empty and closed (by continuity). We infer $A = D$ hence h is constant in D. □

2.4. The Dirichlet problem in the disc. Let $\Omega \subset\subset \mathbb{C}$ be a bounded domain and $\phi : \partial\Omega \to \mathbb{R}$ a continuous function (the boundary data). The Dirichlet problem for the homogeneous Laplace equation consists in finding a harmonic function $h : \Omega \to \mathbb{R}$ solution of the following linear PDE with prescribed boundary values,

$$\mathrm{DP}(\Omega, \phi, \mathbf{0}) \quad \begin{cases} \Delta h = \mathbf{0} \text{ in } \Omega \\ h_{|\partial\Omega} = \phi \end{cases}$$

By the maximum principle, if a solution exists it is unique. We only treat here the case when Ω is the unit disc

$$\mathbb{D} := \{\zeta \in \mathbb{C}\,;\, |\zeta| < 1\}.$$

The solution can be expressed by using the Poisson transform:

Proposition 1.6. *Assume* $\phi \in \mathcal{C}^0(\partial\mathbb{D})$. *The function*

$$z \mapsto h_\phi(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\phi(e^{i\theta})d\theta$$

is harmonic in $\mathbb{D}$ *and continuous up to the boundary where it coincides with* ϕ. *Thus* h_ϕ *is the unique solution to* $\mathrm{DP}(\Omega, \phi, \mathbf{0})$.

Proof. Observe that for fixed $\zeta = e^{i\theta} \in \partial\mathbb{D}$, the Poisson kernel is the real part of a holomorphic function in $\mathbb{D}$,

$$P_{\mathbb{D}}(\zeta, z) := \frac{1}{2\pi}\frac{1-|z|^2}{|z-\zeta|^2} = \frac{1}{2\pi}\Re\left(\frac{\zeta+z}{z-\zeta}\right).$$

Thus h_ϕ is harmonic in $\mathbb{D}$ as an average of harmonic functions.

We now establish the continuity property. Fix $\zeta_0 = e^{i\theta_0} \in \partial\mathbb{D}$ and $\varepsilon > 0$. Since ϕ is continuous at ζ_0, we can find $\delta > 0$ such that $|\phi(\zeta) - \phi(\zeta_0)| < \varepsilon/2$ whenever $\zeta \in \partial\mathbb{D}$ and $|\zeta - \zeta_0| < \delta$. Observing that the Poisson formula for $h \equiv 1$ implies

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2}d\theta \equiv 1,$$

we infer

$$|h_\phi(z) - \phi(\zeta_0)| \le \varepsilon/2 + M\int_{|e^{i\theta}-\zeta_0|\ge\delta} \frac{1-|z|^2}{|z-e^{i\theta}|^2}d\theta,$$

where $2\pi M = \sup_{\partial\mathbb{D}}|\phi|$. Note that $|z - e^{i\theta}| \ge \delta/2$ if z is close enough to ζ_0 and $|e^{i\theta} - \zeta_0| \ge \delta$. The latter integral is therefore bounded from above by $4(1-|z|^2)/\delta^2$ hence converges to zero as z approaches the unit circle. □

3. Subharmonic functions

We now recall some basic facts concerning subharmonic functions in $\mathbb{R}^2 \simeq \mathbb{C}$. These are characterized by submean-value inequalities.

3.1. Definitions and basic properties. Let $\Omega \subset \mathbb{C}$ be a domain.

Definition 1.7. *A function* $u : \Omega \longrightarrow [-\infty, +\infty[$ *is subharmonic if it is upper semi-continuous in* Ω *and for all* $a \in \Omega$ *there exists* $0 < \rho(a) < \mathrm{dist}(a, \partial\Omega)$ *such that for all* $0 < r < \rho(a)$,

$$u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta. \tag{3.1}$$

Recall that a function u is upper semi-continuous (u.s.c. for short) in Ω if and only if for all $c \in \mathbb{R}$ the sublevel set $\{u < c\}$ in an open subset of Ω. Note that harmonic functions are subharmonic; the class of subharmonic functions is however much larger.

The notion of subharmonicity is a local concept. By semi-continuity, a subharmonic function is bounded from above on any compact subset $K \subset \Omega$ and attains its maximum on K. It can however take the value $-\infty$ at some points. With our definition the function which is identically $-\infty$ is subharmonic in Ω.

We will soon show that if u is subharmonic in a domain Ω and $u \not\equiv -\infty$, then $u \in L^1_{loc}(\Omega)$ hence the set $\{u = -\infty\}$ has zero Lebesgue measure in $\mathbb{C}$. It is called the *polar set* of u.

Observe that the maximum of two subharmonic functions is subharmonic; so is a convex combination of subharmonic functions. Here are some further recipes to construct subharmonic functions:

PROPOSITION 1.8. *Let $\Omega \subset \mathbb{C}$ be a domain in $\mathbb{C}$.*

(1) *If $u : \Omega \longrightarrow [-\infty, +\infty[$ is subharmonic in Ω and $\chi : I \to \mathbb{R}$ is a convex increasing function on an interval I containing $u(\Omega)$ then $\chi \circ u$ is subharmonic in Ω.*
(2) *Let $(u_j)_{j\in\mathbb{N}}$ be a decreasing sequence of subharmonic functions in Ω. Then $u := \lim u_j$ is subharmonic in Ω.*
(3) *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of subharmonic functions in Ω, which is locally bounded from above in Ω and $(\varepsilon_j) \in \mathbb{R}_+^{\mathbb{N}}$ be such that $\sum_{j\in\mathbb{N}} \varepsilon_j < +\infty$. Then $u := \sum_{j\in\mathbb{N}} \varepsilon_j u_j$ is subharmonic in Ω.*
(4) *Let $(X, \mathcal{T})$ be a measurable space, μ a positive measure on $(X, \mathcal{T})$, and $E(z, x) : \Omega \times X \longrightarrow \mathbb{R} \cup \{-\infty\}$ a function s.t.*
   (i) for μ-a.e. $x \in X$, $z \longmapsto E(z, x)$ is subharmonic in Ω,
   (ii) For all $z_0 \in \Omega$, $\exists D$ a neighborhood of z_0 in Ω and $g \in L^1(\mu)$ s.t. $E(z, x) \leq g(x)$ for all $z \in D$ and μ-a.e. $x \in X$.
   Then $z \mapsto U(z) := \int_X E(z, x) d\mu(x)$ is subharmonic in Ω.

PROOF. 1. The first property is an immediate consequence of Jensen's convexity inequality.

2. It is clear that $u = \inf\{u_j; j \in \mathbb{N}\}$ is usc in Ω. The submean-value inequality is a conseqence of the monotone convergence theorem.

3. The statement is local hence it is enough to prove that u is subharmonic in any subdomain $D \Subset \Omega$. By assumption there exists $C > 0$ such that $\sup_D u_j \leq C$ for all $j \in \mathbb{N}$. Write

$$u = \sum_{j\in\mathbb{N}} \varepsilon_j (u_j - C) + C \sum_{j\in\mathbb{N}} \varepsilon_j.$$

The first sum is the limit of a decreasing sequence of subharmonic functions, hence it is subharmonic and so is u.

4. The upper semi-continuity of U is a consequence of Fatou's lemma. The submean value property is a consequence of the Tonelli-Fubini theorem. □

We now give some examples of subharmonic functions.

EXAMPLES 1.9.

1. Fix $a \in \mathbb{C}$ and $c > 0$. The function $z \mapsto c \log |z - a|$ is subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus \{a\}$.

2. Let $(a_j) \in \mathbb{C}^{\mathbb{N}}$ be a bounded sequence and let $\varepsilon_j > 0$ be positive reals such that $\sum_j \varepsilon_j < +\infty$. The function

$$z \mapsto u(z) := \sum_j \varepsilon_j \log |z - a_j|$$

is a locally integrable subharmonic function in $\mathbb{C}$. If the sequence (a_j) is dense in a domain Ω, it follows from a Baire category argument that the polar set $(u = -\infty)$ is uncountable but has zero Lebesgue measure. Any point $z \in \Omega \setminus \{u = -\infty\}$ is a point of discontinuity of u: the function u is finite at z but not locally bounded near z.

We generalize the first example above:

Proposition 1.10. *Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function with $f \not\equiv 0$ in Ω. Then $\log |f|$ is a subharmonic function in Ω which is harmonic in the domain $\Omega \setminus f^{-1}(0)$. In particular for any $\alpha > 0$ the function $|f|^\alpha$ is a subharmonic function in Ω.*

Proof. Observe that $\{u = -\infty\} = \{f = 0\}$. It is clear that u is u.s.c. in Ω, since for every $c \in \mathbb{R}$ $\{u < c\} = \{|f| < e^c\}$ is open.

If $a \in \Omega$ and $u(a) = -\infty$, the submean-value inequality (3.1) is trivially satisfied. If $a \in \Omega$ and $u(a) > -\infty$ then $f(a) \neq 0$. By continuity, $f(z) \neq 0$ for $|z - a| < r$, where $r > 0$ is small enough. It follows that $\log f$ has a continuous branch which is holomorphic in the disc $\mathbb{D}(a, r)$. Therefore $u = \Re(\log f)$ is harmonic in $\mathbb{D}(a, r)$ hence it satisfies the submean-value equality.

The last statement follows from the fact that $|f|^\alpha = \chi(\log |f|)$ where $\chi(t) := \exp(\alpha t)$ is a convex increasing function in $\mathbb{R} \cup \{-\infty\}$. $\square$

There are other connections between convexity and subharmonicity.

Proposition 1.11. *Let $u : \Omega \longrightarrow \mathbb{R}$ be a locally convex function. Then u is a continuous subharmonic function in Ω.*

2. Let $u : \Omega = D \times G \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function which only depends on the real part of z i.e. $u(z) = v(x)$ for any $z = x + iy \in \Omega$, then u is subharmonic in Ω iff v is convex in D,

3. Let u be a radial subharmonic function definid on a disc $\mathbb{D}(0, R)$ i.e. $u(z) := \psi(|z|)$. Then u is subharmonic in $\mathbb{D}(0, R)$ if and only if the function ψ is a convex function of $\log |z|$ in $[-\infty, \log R[$.

We leave the proofs for the reader as an exercise (see Exercise 1.7).

Remark 1.12. *Observe that convex functions are continuous but this is not the case for subharmonic functions as Examples 1.9 show. This is an important source of difficulty when studying fine properties of subharmonic functions.*

The converse of the above proposition is not true: the function $u(x, y) = x^2 - y^2$ is harmonic but not convex.

The mean value of a subharmonic function has an important monotonicity property:

PROPOSITION 1.13. *Let u be a subharmonic function, $a \in \Omega$ and set $\delta(a) := dist(a, \partial\Omega)$. The mean-value*

$$r \longmapsto M(a,r) := \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta,$$

is increasing and continuous in $[0, \delta(a)[$; it converges to $u(a)$ as $r \to 0$.

PROOF. Fix $0 < r < \delta(a)$ and let h be a continuous function in the unit circle $\partial\mathbb{D}$ such that $u(a + re^{i\theta}) \leq h(e^{i\theta})$ for all $e^{i\theta} \in \partial\mathbb{D}$. Let H be the unique harmonic function in $\mathbb{D}$ such that $H = h$ on $\partial\mathbb{D}$. The classical maximum principle insures $u(a + r\zeta) \leq H(\zeta)$ for $\zeta \in \mathbb{D}$.

If $0 < s < r$, it follows from the mean-value property for harmonic functions that

$$\int_0^{2\pi} u(a + se^{i\theta}) d\theta \leq \int_0^{2\pi} H(se^{i\theta}) d\theta = \int_0^{2\pi} H(re^{i\theta}) d\theta.$$

Therefore $\int_0^{2\pi} u(a+se^{i\theta}) d\theta \leq \int_0^{2\pi} h(re^{i\theta}) d\theta$ for any continuous function h such that $u(a + r\zeta) \leq h(\zeta)$ on $\partial\mathbb{D}$.

Since u is upper semi-continuous, there exists a decreasing sequence h_j of continuous functions in the circle $\partial\mathbb{D}$ that converges to the function $\zeta \longmapsto u(a + r\zeta)$ in the circle (see Exercise 1.1). The monotone convergence theorem yields

$$\int_0^{2\pi} u(a + se^{i\theta}) d\theta \leq \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

□

COROLLARY 1.14. *If u is subharmonic in Ω, $a \in \Omega$ and $0 < r < \delta(a)$, then*

$$u(a) \leq \frac{1}{\pi r^2} \int_{\mathbb{D}(a,r)} u(z) d\lambda_2(z),$$

where λ_2 is the Lebesgue measure on $\mathbb{R}^2$. For any $a \in \Omega$,

$$u(a) = \lim_{r \to 0^+} \frac{1}{\pi r^2} \int_{\mathbb{D}(a,r)} u(z) d\lambda_2(z).$$

In particular if u and v are subharmonic functions in Ω such that $u \leq v$ almost everywhere in Ω then $u \leq v$ everywhere in Ω.

This will impy integrability of subharmonic functions.

The submean-value inequalities imply the following important integrability result:

PROPOSITION 1.15.

$$SH(\Omega) \subset L^1_{loc}(\Omega).$$

Moreover the restriction of $u \in SH(\Omega)$ *to any circle* $\partial\mathbb{D}(a,r)$ *such that* $\bar{\mathbb{D}}(a,r) \subset \Omega$ *is integrable with respect to the lenght measure of the circle.*

In particular the polar set $P(u) := \{u = -\infty\}$ *has zero arear in* Ω *and its intersection with any circle has lenght zero.*

Definition 1.16. *A set is called (locally) polar if it is (locally) included in the polar set* $\{u = -\infty\}$ *of a function* $u \in SH(\Omega)$.

It follows from previous proposition that ipolar sets are somehow small. We will provide more precise information on their size in the next chapters.

Proof. Fix $u \in SH(\Omega)$ et let G denote the set of points $a \in \Omega$ such that u is integrable in a neighborhood of a. We are going to show that G is a non empty open and closed subset of Ω. It will follow that $G = \Omega$ (by connectedness) and then $u \in L^1_{loc}(\Omega)$.

Note that G is open by definition. If $a \in \Omega$ and $u(a) > -\infty$, the area submean-value inequalities yield, for all $0 < r < \mathrm{dist}(a, \partial\Omega)$,

$$-\infty < \pi r^{2n} u(a) \leq \int_{\mathbb{B}(a,r)} u(z)\, dV(z).$$

Since u is bounded from above on $\mathbb{B}(a,r) \Subset \Omega$, it follows that u is integrable on $\mathbb{B}(a,r)$. In particular if $u(a) > -\infty$ then $a \in G$, hence $G \neq \emptyset$, since $u \not\equiv -\infty$.

We finally prove that G is closed. Let $b \in \Omega$ be a point in the closure of G and $r > 0$ so that $\mathbb{B}(b,r) \Subset \Omega$. By definition there exists $a \in G \cap \mathbb{B}(b,r)$. Since u is locally integrable in a neighborhood of a there exists a point a' close to a in $\mathbb{B}(b,r)$ such that $u(a') > -\infty$. Since $b \in \mathbb{B}(a',r) \Subset \Omega$ and u is integrable on $\mathbb{B}(a',r)$, it follows that $b \in G$.

The other properties are proved similarly, replacing superficial submean inequalities by circular ones. □

We end this section by proving the following removable singularity theorem.

Proposition 1.17. *Let* $w : \Omega \longrightarrow [-\infty, +\infty[$ *be subharmonic function in* Ω *such that* e^w *is continuous in* Ω *and* $F := \{z \in \Omega; w(z) = -\infty\}$ *its closed polar set. Let* $u : \Omega \setminus F \longrightarrow [-\infty, +\infty[$ *be a subhamonic function which is locally upper bounded near the point in* F*. Then* u *extends uniquely as a subharmonic function in* Ω.

Proof. The hypothesis means that for any $a \in F$, there exsits a disc $D \Subset \Omega$ and a constant $M = M_D$ such that $u \leq M$ in $D \setminus F$. Therefore the function u^* is well defined and is the unique upper semi-continuous extension of u to Ω. To prove the theorem it is enough to show that u^* is subharmonic in each disc $D \Subset \Omega$. For $\varepsilon > 0$, define the following function $v_\varepsilon := u^* + \varepsilon w$ in Ω. Then by definition, u_ε is upper semi-continuous in Ω. It is easy to see that v_ε is subharmonic in Ω, since $v_\varepsilon = u + \varepsilon w$ in $\Omega \setminus F$ and then subharmonic there and the submean value property is trivially satisfied at points in F since $v_\varepsilon = -\infty$ in F.

We claim that v_ε converges pointwise to a subharmonic function v in Ω. Indeed fix a disc $D = \mathbb{D}(a,r) \Subset \Omega$ and let $M := \max_{\bar{D}} w$. Then for any $\varepsilon > 0$, we have

$$v_\varepsilon = u^* + \varepsilon(w - M) + \varepsilon M.$$

It is clear that the family $u^* + \varepsilon(w - M)$ decreases as ε dcereases to 0 so it converges to a function v_D which is then subharmonic in D. Therefore $v_\varepsilon \to v_D$ poinwise in D and $v_D = u$ in $D \setminus F$.

It is then easy to see that all the local subharmonic extensions v_D glue into a unique subharmonic function v such that $v = u$ in $\Omega \setminus F$. $\square$

3.2. The maximum principle. The maximum principle is one of the most powerful tools in Potential Theory:

THEOREM 1.18. *Assume u is subharmonic in Ω.*

1. If u admits a local maximum at some point $a \in \Omega$ then u is constant in a neighborhood of a,

2. For any bounded subdomain $D \Subset \Omega$ we have

$$\max_{\bar{D}} u = \max_{\partial D} u.$$

Moreover $u(z) < \max_{\partial D} u$ for all $z \in D$ unless u is constant on D.

PROOF. The proof follows the same lines as in the case of harmonic functions with some modifications due to the fact that u is not necessarily continuous.

1. By hypothesis there is a disc $\mathbb{D}(a,r) \Subset \Omega$ such that $u(z) \leq u(a)$ for any $z \in \mathbb{D}(a,r)$. Fix $0 < s \leq r$ and observe that

$$u(a) - u(a + se^{i\theta}) \geq 0 \text{ for all } \theta \in [0, 2\pi].$$

Integrating in polar coordinates gives

$$\int_{\mathbb{D}(a,r)} (u(a) - u(z)) dV(z) \geq 0,$$

while the submean-value property shows that the above integral is negative. Therefore

$$\int_{\mathbb{D}(a,r)} (u(z) - u(a)) dV(z) = 0.$$

We infer that $u(z) - u(a) = 0$ almost everywhere in $\mathbb{D}(a,r)$, hence everywhere in $\mathbb{D}(a,r)$ since u is subharmonic. This proves that u is constant in a neighborhood of a.

2. By compactness and upper semi-continuity we can find $a \in \bar{D}$ such that $u(a) = \max_{\bar{D}} u$. If $a \in D$ then by the previous case u is

constant in a neighborhood of a, therefore the set

$$A := \{z \in D; u(z) = u(a) = \max_{\bar{D}} u\} = \{z \in D; u(z) \geq \max_{\bar{D}} u\}$$

is open, non empty and closed by upper semi-continuity, hence u is constant in D. □

COROLLARY 1.19. *Let $\Omega \Subset \mathbb{C}$ a bounded domain and u a subharmonic function in Ω. Assume that $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial\Omega$. Then $u \leq 0$ in Ω.*

PROOF. Fix $\varepsilon > 0$. By compactness and upper semi-continuity of u there exists a compact subset $K \subset \Omega$ such that $u \leq \varepsilon$ in $\Omega \setminus K$. Take a subdomain $D \Subset \Omega$ such that $K \subset D$ and apply Theorem 1.18 to conclude that $u \leq \varepsilon$ in D. Therefore $u \leq \varepsilon$ in Ω and the conclusion follows since $\varepsilon > 0$ is arbitrary. □

The following consequence is known as the *comparison principle*:

COROLLARY 1.20. *Let $\Omega \Subset \mathbb{C}$ be a bounded domain and u, v subharmonic functions in $L^1_{loc}(\Omega)$ such that the following holds:*
(i) For all $\zeta \in \partial\Omega$, $\liminf_{z\to\zeta}(u(z) - v(z)) \geq 0$.
(ii) $\Delta u \leq \Delta v$ in the weak sense of distributions in Ω.
Then $u \geq v$ in Ω.

PROOF. Set $w := v - u$ and observe that w is well defined at all points in Ω where the two functions do not take the value $-\infty$ at the same time, hence almost everywhere in Ω and $w \in L^1_{loc}(\Omega)$ (see Proposition 2.8). From the condition (ii) it follows that $\Delta w \geq 0$ in the sense of distributions in Ω.

We infer that w is equal almost everywhere to a subharmonic function W in Ω (see Proposition 1.25). Therefore $v = u + W$ almost everywhere in Ω, hence everywhere in Ω.

We claim that $\limsup_{z\to\zeta} W(z) \leq 0$. Indeed let $(z_j) \in \Omega^{\mathbb{N}}$ be a sequence converging to $\zeta \in \partial\Omega$. Since $\limsup_{z_j\to\zeta}(v(z_j) - u(z_j)) \leq 0$, it follows that for $j > 1$ large enough $v(z_j) - u(z_j) < +\infty$, hence $W(z_j) = v(z_j) - u(z_j)$ and $\limsup_{z_j\to\zeta} W(z_j) \leq 0$. This proves our claim. It follows from Corollary 1.19 that $W \leq 0$ in Ω as desired. □

3.3. Maximal subharmonic functions. We will show that harmonic functions can be characterized among subharmonic functions by the maximum principle.

We first prove the following result.

PROPOSITION 1.21. *Let $u : \bar{\mathbb{D}} \longrightarrow [-\infty, +\infty[$ be an upper semi-continuous in $\bar{\mathbb{D}}$ which is subharmonic in $\mathbb{D}$. Define the Poisson transform of u in $\mathbb{D}$ by the following formula:*

$$P_{\mathbb{D}} u(z) := \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta.$$

Then $P_{\mathbb{D}}u$ satisfies the following properties:

1. $P_{\mathbb{D}}u$ is harmonic in $\mathbb{D}$, $u \leq P_{\mathbb{D}}u$ in $\mathbb{D}$ and for any $\zeta \in \partial\mathbb{D}$,

$$\limsup_{\mathbb{D}\ni z\to\zeta} h(z) \leq u(\zeta);$$

2. for any harmonic function g in $\mathbb{D}$ such that $u \leq g$ in $\mathbb{D}$, we have $P_{\mathbb{D}}u \leq g$ in $\mathbb{D}$ i.e. $h := P_{\mathbb{D}}u$ is the smallest harmonic majorant of u in $\mathbb{D}$;

Proof. 1. Since the Poisson kernel is harmonic in $z \in \mathbb{D}$ when $\zeta \in \partial\mathbb{D}$, it follows that $h := P_D u$ is harmonic in $\mathbb{D}$. Let $(\phi_j)_{j\in\mathbb{N}}$ be a decreasing sequence of continuous functions in $\partial\mathbb{D}$ converging to u. It follows from the previous result that $h_j := P_D\phi_j$ is harmonic and $h_j = \phi_j$ in $\partial\mathbb{D}$. By the maximum principle we have $u \leq h_j$, which implies that $u \leq P_D u$ by monotone convergence theorem. On the other hand by definition we have $h = P_{\mathbb{D}}u \leq h_j$ in $\mathbb{D}$. Hence for any $\zeta \in \partial\mathbb{D}$, $\limsup_{\mathbb{D}\ni z\to\zeta} h(z) \leq \phi_j(\zeta)$. Now letting $j \to +\infty$, we obtain $\limsup_{\mathbb{D}\ni z\to\zeta} h(z) \leq u(\zeta)$.

Now since $u \leq P_{\mathbb{D}}u = h$ in $\mathbb{D}$ it follows that 2. By scaling the functions we can assume that they are all defined in a neighborhood of the closed disc $\bar{\mathbb{D}}$. Then the statement follows from Proposition 1.6. □

Proposition 1.22. *1. Let $u : \Omega[-\infty, +\infty[$ be a subharmonic function in a domain $\Omega \subset \mathbb{C}$. Then for any subdomain $G \Subset \Omega$ and any continuous function $h : \bar{G} \longrightarrow \mathbb{R}$ which is harmonic in G such that $\limsup_{z\to\zeta} u(z) \leq h(\zeta)$ for all $\zeta \in \partial G$, we have $u \leq h$ in G.*

2. Conversely let $w : \Omega \longrightarrow \mathbb{R}$ be a subharmonic function in a domain $\Omega \subset \mathbb{C}$ which is maximal in the following sense: for any disc $\bar{\mathbb{D}}(a, r) \subset \Omega$ and any subharmonic function v in $\mathbb{D}(a, r)$, the inequality $v^* \leq w$ in $\partial\mathbb{D}(a, r)$ implies $v \leq w$ in $\mathbb{D}(a, r)$. Then w is harmonic in Ω.

Proof. 1. The first statement follows from the classical maximum principle applied to the subharmonic function $u - h$ in G.

2. It is enough to prove that w is harmonic in each disc $\mathbb{D}(a, r) \Subset \Omega$. Let $h := P_{\mathbb{D}(a,r)}w$ be the Poisson transform of w in the disc $\mathbb{D}(a, r)$. Then h is harmonic in $\mathbb{D}(a, r)$ and $w \leq h$ in $\mathbb{D}(a, r)$. On the other hand we also know that $h^* \leq w$ in $\partial\mathbb{D}(a, r)$. Hence by maximality, we conclude that $h \leq w$ in $\mathbb{D}(a, r)$. Thus we obtain the equality $h = w$ which proves that w is harmonic in $\mathbb{D}(a, r)$. □

3.4. Poisson modification, balayage. We show the following result which is useful when solving the Dirichlet problem. This called the balayage procedure.

Proposition 1.23. *Let $u : \Omega \longrightarrow [-\infty, +\infty[$ be a subharmonic function and let $D \Subset \Omega$ be a disc. Then the function $\tilde{u}$ defined in Ω by*

$$\tilde{u} : P_D u \text{ in } D, \ \ \tilde{u} = u \text{ in } \Omega \setminus D,$$

satisfies the following properties:

1. $u \leq \tilde{u}$ *in* Ω *and* $\tilde{u} = u$ *in* $\Omega \setminus D$*;*
2. $\tilde{u}$ *is subharmonic in* Ω *and harmonic in* D.

PROOF. By the previous result we have that $u \leq \tilde{u}$ in D. Then it follows that $\tilde{u}$ is upper semi-continuous at boundary points $\zeta \in \partial D$ and satisfies the submean value inequalities at that point, hence it is subharmonic in Ω. □

4. Riesz representation formulas

In this section we lay down the foundations of *Logarithmic potential theory*. We associate a canonical (Riesz) measure to any subharmonic function and show how to reconstruct the function from its boundary values and its Riesz measure.

4.1. The logarithmic potential.

DEFINITION 1.24. *We let* $SH(\Omega)$ *denote the set of all subharmonic functions in the domain* Ω *which are not identically* $-\infty$.

We have seen that the set $SH(\Omega)$ is a convex positive cone contained in $L^1_{loc}(\Omega)$.

PROPOSITION 1.25. *If* $u \in SH(\Omega)$ *then the distribution* $\Delta u \geq 0$ *is a non-negative distribution: for any positive test function* $\varphi \in \mathcal{D}^+(\Omega)$,

$$\langle \Delta u, \varphi \rangle = \int_\Omega u \Delta \varphi \, dV \geq 0.$$

Conversely if $T \in \mathcal{D}'(\Omega)$ *is a distribution such that* $\Delta T \geq 0$ *then there is a unique function* $u \in SH(\Omega)$ *such that* $T_u = T$.

PROOF. Fix $u \in SH(\Omega)$. Assume first that u is smooth in Ω and fix $a \in \Omega$. It follows from Taylor's formula that

$$\Delta u(a) = \lim_{r \to 0^+} \frac{2}{r^2} \left(\frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta - u(a) \right).$$

Since u is subharmonic the right hand side is non negative hence $\Delta u(a) \geq 0$ pointwise in Ω.

We now get rid of the regularity assumption. It follows from Proposition 2.8 that $u \in L^1_{loc}(\Omega)$, hence we can regularize u by convolution setting $u_\varepsilon = u \star \rho_\varepsilon$ for $\varepsilon > 0$, using radial mollifiers.

The functions u_ε are subharmonic as convex combination of subharmonic functions. Since u_ε is moreover smooth we infer $\Delta u_\varepsilon \geq 0$, hence $\Delta u \geq 0$ since $u_\varepsilon \to u$ in L^1_{loc}.

We note for later use that $\varepsilon \mapsto u_\varepsilon$ is non-decreasing: this follows from the mean value inequalities and the fact that we use radial and non-negative mollifiers. In particular u_ε decreases to u as ε decreases to zero (cf Proposition 1.13 and Corollary 1.14).

Let now T be a distribution in Ω and $(\rho_\varepsilon)_{\varepsilon>0}$ be mollifiers as above. Then $v_\varepsilon := T \star \rho_\varepsilon$ is a smooth function such that $\Delta v_\varepsilon = (\Delta T) \star \rho_\varepsilon \geq 0$ in Ω_ε, thus v_ε is subharmonic in Ω_ε.

We claim that $\varepsilon \longmapsto v_\varepsilon$ is non decreasing. Indeed for $\varepsilon > 0$ small enough, the map $\eta \longmapsto (v_\varepsilon \star \rho_\eta)$ is non decreasing since v_ε is subharmonic in Ω_ε. By definition of convolution, we have $v_\varepsilon \star \rho_\eta = v_\eta \star \rho_\varepsilon$ in $\Omega_{\varepsilon+\eta}$ for $\varepsilon, \eta > 0$ small enough. Therefore for any fixed $\eta > 0$ small enough, the map $\varepsilon \longmapsto v_\varepsilon \star \rho_\eta$ is also non decreasing for small ε. Since $v_\varepsilon \star \rho_\eta \to v_\varepsilon$ as $\eta \to 0^+$ the claim follows.

Now since v_ε is non decreasing in $\varepsilon > 0$ it converges to a subharmonic function u as ε decreases to zero. The function u can not be identically $-\infty$ since $T_u = T$ as distributions (this follows from the monotone convergence theorem). The uniqueness follows again from Corollary 1.14: two subharmonic functions which coincide almost everywhere are actually equal. □

Recall that a positive distribution always extends to a positive Borel measure (see Exercise 1.12). Therefore if $u \in SH(\Omega)$ then the positive distribution $(1/2\pi)\Delta u$ can be extended as a positive Borel measure μ_u on Ω which we call the Riesz measure of u.

Definition 1.26. *The Riesz measure of $u \in SH(\Omega)$ is*

$$\mu_u = \frac{1}{2\pi}\Delta u.$$

Using the complex coordinate $z = x + iy$, the real differential operator acting on smooth functions $f : \Omega \longrightarrow \mathbb{C}$ by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

splits into $d = \partial + \overline{\partial}$, where the complex differential operators ∂ and $\overline{\partial}$ are defined by

$$\partial f = \frac{\partial f}{\partial z}dz \quad \text{and} \quad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}}d\overline{z}.$$

We extend these differential operators to distributions: if f is a distribution then df is defined as above, but it has to be understood in the sense of currents of degree 1 on Ω: it is a differential form of degree 1 with distribution coefficients (see next chapter).

Observe that the volume form in $\mathbb{C}$ can be writen as

$$dx \wedge dy = \frac{i}{2} dz \wedge d\overline{z}.$$

We define the real operator d^c by

$$d^c := \frac{1}{2i\pi}(\partial - \overline{\partial})$$

so that for $u \in SH(\Omega)$, we obtain

$$dd^c u = \frac{1}{2\pi}\Delta u \, dx \wedge dy = \mu_u \, dx \wedge dy,$$

where μ_u is the Riesz measure of u and the notation $\mu_u\, dx \wedge dy$ is understood in the sense of currents of degree 2: it is a differential form of degree 2 with distribution coefficients.

EXAMPLE 1.27. *Fix $a \in \Omega$. The function $z \mapsto \ell_a(z) := \log|z-a|$ is subharmonic and satisfies*

$$dd^c \ell_a = \frac{1}{2\pi}\Delta \ell_a = \delta_a, \tag{4.1}$$

in the sense of distribution, where δ_a denotes the Dirac mass at the point a. In particular ℓ_0 is a fundamental solution of the linear differential operator $dd^c = (2\pi)^{-1}\Delta$ in $\mathbb{C}$.

The following result connects logarithmic potential theory to the theory of holomorphic functions in one complex variable:

PROPOSITION 1.28. *Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function such that $f \not\equiv 0$, then $\log|f| \in SH(\Omega)$. It satisfies*

$$dd^c \log|f| = \sum_{a \in Z_f} m_f(a)\delta_a,$$

where $Z_f := f^{-1}(0)$ is the zero set of f in Ω and $m_f(a)$ is the order of vanishing of f at the point a.

Observe that since $f \not\equiv 0$, the zero set Z_f is discrete in Ω, hence the sum is locally finite.

PROOF. On any subdomain $D \Subset \Omega$ the set $A := Z_f \cap D$ is finite and there exists a non-vanishing holomorphic function g such that

$$f(z) = \Pi_{a\in A}(z-a)^{m_f(a)} g(z)$$

for $z \in D$. Since g is zero free, $\log|g|$ is harmonic hence

$$dd^c \log|f| = \sum_{a\in A} m_f(a) dd^c \log|z-a| = \sum_{a \in A} m_f(a)\delta_a$$

in the sense of distributions. □

Let μ be a Borel measure with compact support on $\mathbb{C}$, then

$$z \mapsto U_\mu(z) := \int_{\mathbb{C}} \log|z-\zeta| d\mu(\zeta) = \mu \star \ell_0(z)$$

is subharmonic in $\mathbb{C}$. Moreover, if $z \in \mathbb{C} \setminus \operatorname{Supp}(\mu)$ then

$$U_\mu(z) \geq \log \operatorname{dist}(z, \operatorname{Supp}(\mu)) > -\infty,$$

and

$$U_\mu(z) \leq \mu(\mathbb{C}) \log^+|z| + C(\mu),\ z \in \mathbb{C}.$$

It follows that $U_\mu \in SH(\mathbb{C})$.

DEFINITION 1.29. *The function $U_\mu : z \mapsto \int_{\mathbb{C}} \log|z-\zeta| d\mu(\zeta)$ is called the logarithmic potential of the measure μ.*

Observe that

$$\frac{1}{2\pi}\Delta U_\mu = \left(\frac{1}{2\pi}\Delta \ell_0\right) \star \mu = \mu,$$

in the sense of distributions on $\mathbb{C}$. This implies that U_μ is subharmonic in $\mathbb{C}$ and harmonic (hence real analytic) in $\mathbb{C} \setminus \mathrm{Supp}(\mu)$.

4.2. The Riesz decomposition Formula. We can now derive the *Riesz decomposition formula*:

PROPOSITION 1.30. *Fix $u \in SH(\Omega)$ and $D \Subset \Omega$ a subdomain. Then*

$$u(z) = \int_D \log|z-\zeta| d\mu_u(\zeta) + h_D(z), \quad z \in D,$$

where $\mu_u := \frac{1}{2\pi}\Delta u$ and h_D is a harmonic function in D.

PROOF. Apply the last construction to the measure $\mu_D := \mathbf{1}_D \cdot \mu_u$ which is a Borel measure with compact support on $\mathbb{C}$: the function

$$v(z) := \int_D \log|z-\zeta| d\mu_u(\zeta) = \mu_D \star \ell_0(z)$$

is subharmonic in $\mathbb{C}$ and satisfies

$$\Delta v = \mu_D \star \Delta(\ell_0) = \mathbf{1}_D(\Delta u)$$

in the sense of Borel measures in $\mathbb{C}$. Therefore $h := u - v$ is a locally integrable function which satisfies $\Delta h = 0$ in the weak sense of distributions in D. It follows from Weyl's lemma that h coincides almost everywhere in D with a harmonic function denoted by h_D. This implies that $u = v + h_D$ almost everywhere in D hence everywhere in D. ☐

This result shows that a subharmonic function u coincides locally (up to a harmonic function which is smooth) with the logarithmic potential of its *Riesz measure* μ_u. In particular the information on the singularities of u (discontinuities, polar points, etc) are contained within its potential U_μ.

The study of fine properties of subharmonic functions is therefore reduced to that of logarithmic potentials of compactly supported Borel measures on $\mathbb{C}$, hence the name *Logarithmic Potential Theory*.

4.3. Poisson-Jensen formula. The *Poisson-Jensen formula* is a generalization of the Poisson formula for harmonic functions in the unit disc. It is a precise version of the Riesz representation formula that takes into account the boundary values of the function:

PROPOSITION 1.31. *Let $u : \overline{\mathbb{D}} \longrightarrow [-\infty, +\infty[$ be an upper semi-continuous function which extends as a subharmonic in a neighjbourhhod of $\overline{\mathbb{D}}$. Then for all $z \in \mathbb{D}$,*

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) \frac{1-|z|^2}{|z-e^{i\theta}|^2}\frac{d\theta}{2\pi} + \int_{|\zeta|<1} \log\frac{|z-\zeta|}{|1-z\bar{\zeta}|} d\mu(\zeta),$$

where $\mu = \frac{1}{2\pi}\Delta u$ *is the Riesz measure of* u.

When u is harmonic in $\mathbb{D}$ we recover Poisson formula (Theorem 1.6). The first term is called the *Poisson transform* of u in $\mathbb{D}$. This is at harmonic majorant of u in $\mathbb{D}$. The second term is a non-positive subharmonic function encoding the singularities of u; it is called the *Green potential* of the measure μ.

Proof. Fix $w \in \mathbb{D}$ and set for $z \in \mathbb{D}$,

$$G_{\mathbb{D}}(z,w) = G_w(z) := \log \frac{|z-w|}{|1-z\bar{w}|}$$

Observe that G_w is subharmonic in $\mathbb{D}$, $dd^c G_w = \delta_w$ in the weak sense of distributions in $\mathbb{D}$ and $G_w \leq 0$ in $\mathbb{D}$ with G_w identically 0 on $\partial\mathbb{D}$.

It follows from the comparison principle (Corollary 1.20) that G_w is the unique function having these properties. It is called the Green function of the unit disc with logarithmic singularity at the point w.

For $w \in \mathbb{C}$ fixed, we set

$$H_w(z) := \int_{\partial\mathbb{D}} \log|\xi - w| P_{\mathbb{D}}(z,\xi) d\sigma(\xi),$$

where $d\sigma$ is the normalized Lebesgue measure on $\partial\mathbb{D}$ and

$$P_{\mathbb{D}}(z,\xi) := \frac{1-|z|^2}{|\xi - z|^2},$$

is the Poisson kernel for the unit disc. We claim that

$$H_w(z) = \log|z-w| - G_w(z), \text{ if } w \in \mathbb{D}, \tag{4.2}$$

$$H_w(z) = \log|z-w|, \quad \text{if } w \in \mathbb{C}\setminus\mathbb{D}. \tag{4.3}$$

Indeed if $w \in \mathbb{D}$ then by Proposition 1.6, the function H_w is harmonic in $\mathbb{D}$ and continuous up to the boundary where it coincides with the function $z \longmapsto \log|z-w|$. Therefore the function $g(z) := \log|z-w| - H_w(z)$ is harmonic in $\mathbb{D}\setminus\{w\}$, subharmonic in $\mathbb{D}$ with a logarithmic singularity at w and is 0 on $\partial\mathbb{D}$. The maximum principle thus yields

$$G_w(z) = \log|z-w| - H_w(z), \tag{4.4}$$

for any $z \in \mathbb{D}$, which proves (4.2).

If $w \in \mathbb{C}\setminus\bar{\mathbb{D}}$, the function $z \longmapsto \log|z-w|$ is harmonic in $\mathbb{D}$ and continuous in $\bar{\mathbb{D}}$. Therefore (4.3) follows from the Poisson formula.

By the Riesz representation formula, if $\mathbb{D}'$ is a disc containing $\bar{\mathbb{D}}$ so that u is subharmonic on $\bar{\mathbb{D}}'$, we have $u = U_\mu + h$ in $\mathbb{D}'$, where $\mu := \mu_{\mathbb{D}'}$ and h is a harmonic function in $\mathbb{D}'$. Fubini's theorem and (4.2), (4.3) yield

$$\begin{aligned}\int_{\partial\mathbb{D}} u(\xi)P_{\mathbb{D}}(z,\xi)d\sigma(\xi) &= \int_{\partial\mathbb{D}}\left(\int_{\mathbb{D}'}\log|\xi-\zeta|d\mu(\zeta)+h(\xi)\right)P_{\mathbb{D}}(z,\xi)d\sigma(\xi)\\ &= \int_{\mathbb{D}'}\left(\int_{\partial\mathbb{D}}\log|\xi-\zeta|P_{\mathbb{D}}(z,\xi)d\sigma(\xi)\right)d\mu(\zeta)+h(z)\\ &= \int_{\mathbb{D}'}\log|z-\zeta|d\mu(\zeta)-\int_{\mathbb{D}}G_\zeta(z))d\mu(\zeta)+h(z)\\ &= u(z)-\int_{\mathbb{D}}G_\zeta(z))d\mu(\zeta),\end{aligned}$$

which is the required formula. □

REMARK 1.32. *The Poisson-Jensen formula suggests to consider the following general Dirichlet problem: given a finite Borel measure on the disc* $\mathbb{D}$ *and a continuous function* ϕ *in* $\partial\mathbb{D}$*, find* $u \in SH(\mathbb{D})$ *which extends to the boundary such that*

$$\mathrm{DP}(\mathbb{D},\phi,\mu)\ \begin{cases}\Delta u = \mu \ \textit{in}\ \mathbb{D}\\ u_{|\partial\mathbb{D}} = \phi\end{cases}$$

We shall come back to this in the next section.

5. The classical Dirichlet problem

As we already observed, for any $w \in \mathbb{C}$, the function $\ell_w(z) := \log|z-w|$ is a fundamental solution for the Laplace operator i.e.

$$(1/2\pi)\Delta\,\ell_w = \delta_w,$$

in the weak sense of distributions on $\mathbb{C}$.

The general Dirichlet problem for the Laplace operator can be stated as follows.

Let $\Omega \Subset \mathbb{C}$ be a bounded domain, $\phi : \partial\Omega \longrightarrow \mathbb{R}$ a continuous function and μ a positive Borel measure on Ω. Find a function $U \in SH(\Omega)$ such that

$$\begin{cases}\Delta U = \mu, \ \text{ on } \ \Omega\\ U_{|\partial\Omega} = \phi\end{cases} \tag{5.1}$$

We call this problem the Dirichlet problem $\mathrm{DP}(\Omega,\phi,\mu)$ in Ω with boundray data ϕ and right hand side μ.

It follows from the maximum principle that the solution if it exists is unique. The following observation, known as the superposition principle, will simplify the study of the general Dirichlet problem by splitting it into two problems, the first of which is much easier to treat.

Since the Laplace operator Δ is linear, we can use the superposition method: If $H_{\Omega,\phi}$ is the solution to the homogenuous Dirichlet problem

DP$(\Omega, \phi, \mathbf{0})$ in Ω and $V_{\Omega,\mu}$ is the solution to the non homogeneous Dirichlet problem DP$(\Omega, 0, \mu)$ in Ω then the function

$$U_{\Omega,\phi,\mu} := H_{\Omega,\phi} + V_{\omega,\mu}$$

is the unique solution to the Dirichlet problem DP(Ω, ϕ, μ) in Ω.

5.1. The homogenuous Dirichlet problem. We start with a definition.

DEFINITION 1.33. *Let $\Omega \Subset \mathbb{C}$ be a domain and $\zeta \in \partial\Omega$. A (strong) subbarrier for the Dirichlet problem in Ω at the boundary point ζ is a subharmonic function b_ζ in Ω such that $b_\zeta < 0$ in $\bar{\Omega} \setminus \{\zeta\}$ and $\lim_{z \to \zeta} b_\zeta(z) = 0$. A domain is said to be regular for the classical Dirichlet problem if it admits a subbarrier at each boundary point.*

Observe that any disc $\mathbb{D}(a, r)$ is regular for the Dirichlet problem, while the punctured disc is not by the maximum principle.

Let us prove that we can solve the homogenuous Dirichlet problem with any continuous boundary data when the domain is regular.

Let ϕ be a continuous function in $\partial\Omega$. By the maximum principle, the solution H to the Dirichlet problem DP$(\Omega, \phi, 0)$ is unique, if it exists.

Moreover, assume that the Dirichlet problem DP$(\Omega, \phi, 0)$ has a solution H. We prove that H coincides with the maximal subsolution. Indeed, define a *a subsolution* to the Dirichlet problem DP$(\Omega, \phi, 0)$ to be a function $u \in SH(\Omega)$ which satisfies the boundary condition

$$u^*(\zeta) := \limsup_{z \to \zeta} u(\zeta) \le \phi(\zeta),\ \forall \zeta \in \partial\Omega.$$

and let $\mathcal{S}(\Omega, \phi)$ be the class of subsolutions to DP$(\Omega, \phi, 0)$. Then by definition $H \in\in \mathcal{S}(\Omega, \phi)$, and by the maximum principle, for any $u \in \mathcal{S}(\Omega, \phi)$ we have $u \le H$. Therefore if we can consider the upper envelope of subsolutions defined by the following formula:

$$U_{\Omega,\phi}(z) := \sup\{u(z);\ u \in \mathcal{S}(\Omega, \phi)\} \tag{5.2}$$

it follows that that $U_{\Omega,\phi} = H$ in Ω.

In other words, the solution to DP$(\Omega, \phi, 0)$ if it exists, coincides with the maximal subsolution to DP$(\Omega, \phi, 0)$.

So the stategy for proving the existence of a solution to DP$(\Omega, \phi, 0)$ is to consider the upper envelope of subsolution defined by (5.2). But there are many difficulties to overcome:

- Do a subsolution exists i.e. is $\mathcal{S}(\Omega, \phi) \neq \emptyset$?
- Is the class $\mathcal{S}(\Omega, \phi)$ (locally) upper bounded in Ω ?
- Is the upper envelope $U_{\Omega,\phi}$ a subsolution ?
- Is the maximal subsolution $U_{\Omega,\phi}$ a solution ?

We can prove the main result of this section.

THEOREM 1.34. *Let $\Omega \Subset \mathbb{C}$ be a bounded domain. Then the class $\mathcal{S}(\Omega, \phi)$ is not empty, uniformly bounded in Ω, stable by finite maxima and Poisson modification over any disc $D \Subset \Omega$.*

Moreover if Ω is regular for the Dirichlet problem and ϕ a continuous function in $\partial\Omega$, the maximal subsolution $U_{\Omega,\phi}$ is the unique harmonic function in Ω such that

$$\lim_{z \to \zeta} U_{\Omega,\phi}(z) = \phi(\zeta).$$

In other words $U_{\Omega,\phi}$ is the unique solution to the Dirichlet problem $\mathrm{DP}(\Omega, \phi, 0)$.

PROOF. Indeed the constant fonction $u_0 = m := \min_{\partial\Omega} \phi$ is obviously a subsolution to $\mathrm{DP}(\Omega, \phi, 0)$, hence $u_0 \in \mathcal{S}(\Omega, \phi)$. On the other hand for any $u \in \mathcal{S}(\Omega, \phi)$, we have $u^* \leq \phi \leq \max_{\partial\Omega} =: M$ in $\partial\Omega$, hence by the maximum principle $u \leq M$ in Ω.

It is clear that $\mathcal{S}(\Omega, \phi)$ is stable under finite maxima and Poisson modification over any disc $D \Subset \Omega$. Therfore we can apply the Lemma above to conclude that U^* is harmonic in $\Omega *$. It remains to show that U^* has boudary values equal to ϕ in $\partial\Omega$. It is here where we use the regularity of the domain Ω. Indeed let $\zeta_0 \in \partial\Omega$ be fixed. By regularity, there exists a subbarrier b_0 at the point ζ_0. Since ϕ is continuous at ζ_0, given $\varepsilon > 0$, there exists $\eta > 0$ such that for any $\zeta \in \mathbb{D}(\zeta_0, \eta) \cap \partial\Omega$ we have $\phi(\zeta_0) - \varepsilon \leq \phi(\zeta) \leq \phi(\zeta_0) + \varepsilon$. Let $A > 1$ be large constant to be specified. Since $b_0 < 0$ in $\bar{\Omega} \setminus \{\zeta_0\}$, it follows by compactness that for $A > 1$ large enough, we have

$$Ab_0(\zeta) + f(\zeta_0) - \varepsilon, \forall \zeta \in \partial\Omega \setminus \mathbb{D}(\zeta_0, \eta).$$

Therefore the function $Ab_0 + f(\zeta_0) - \varepsilon \in \mathcal{S}(\Omega, \phi)$ is a continuous subsolution. Hence $Ab_0(\zeta) + \phi(\zeta_0) - \varepsilon \leq U$ in Ω. Taking the limit at ζ_0 from inside Ω and letting $\varepsilon \to 0$, we finally get

$$\phi(\zeta_0) \leq \liminf_{z \to \zeta_0} U(z).$$

The same barrier argument with $-\phi$ instead of ϕ gives a continuous subharmonic function $v := Ab_0 - \phi(\zeta_0) - \varepsilon$ with boundary values less that $-\phi$. By the maximum principle for any subsolution $u \in \mathcal{S}(\Omega, \phi)$, we have $u + v \leq 0$ in Ω, hence $U \leq -v$ in Ω and $U^* |eq - v$ in Ω by continuity of v.. As above this implies that

$$\limsup_{z \to \zeta_0} U^*(z) \leq \phi(\zeta_0).$$

in conclusion for any $\zeta_0 \in \partial\Omega$, $\lim_{z \to \zeta_0} U^*(z) = \phi(\zeta_0)$. This implies that U^* is a subsolution, hence $U^* =\leq U$ in Ω. Therefore U is a subharmonic function with ϕ as boundary values. To prove the theorem, it remains to show that $\Delta U = \mu$ weakly on Ω. To do so we proceed by balayage. Let $D \Subset \Omega$ be a disc. Then by Poisson modifiaction we construct a subahrmonic function $\hat{U} \in SH(\Omega)$ which is harmonic in

D such that $U \leq \hat{U}$ in Ω and $\hat{U} = U$ in $\Omega \setminus D$. Since $\hat{U}$ coincides with U near the boundary of Ω, it follows that $\hat{U} = \phi$ in $\partial\Omega$. Therefore $\hat{U} \in \mathcal{S}(\Omega, \phi)$ is a subsolution, hence $\hat{U} \leq U$. In conclusion we obtain $U = \hat{U}$ in Ω, in particular U is harmonic in D. The theorem is proved. □

5.2. The Green function. For the unit disc $\mathbb{D}$ we have found a fundamental solution $G_{\mathbb{D}}(\cdot, w)$ for the Laplace operator adapted to the domain $\mathbb{D}$ i.e. such that $G_{\mathbb{D}}(\cdot, w) \equiv 0$ on $\partial\mathbb{D}$. This means that $G_{\mathbb{D}}(\cdot, w)$ is the unique solution to the following non homogeneous Dirichlet problem $\mathrm{DP}(\mathbb{D}, \phi, \mu)$ in $\mathbb{D}$ with $\mu = \delta_w$ the Dirac measure δ_w.

$$\begin{cases} (1/2\pi)\Delta v = \delta_w \text{ in } \mathbb{D} \\ v_{|\partial\mathbb{D}} = 0 \end{cases} \tag{5.3}$$

This suggests the following general definition.

Definition 1.35. *Let $\Omega \subset \mathbb{C}$ be a domain and $\xi \in \mathbb{D}$ be a fixed point. We say that Ω admits a Green function with logarithmic pole at ξ if there exists a fundamental solution to the Laplace operator on the domain Ω with boundary values* 0 *(on $\partial\Omega$) i.e. the Dirichlet problem* $\mathrm{DP}(\Omega, 0, \delta_{\mathrm{w}})$ *with $\mathbb{D}$ replaced by Ω has a solution.*

Such a function if it exists is unique by the maximum principle. We denote it by $G_\Omega(\cdot, \xi)$: the Green function of Ω with logarithmic pole at ξ. The function G_Ω is called the Green kernel.

The existence of a Green function is closely related to the regularity of the domain and the solvability of the Dirichlet problem for the homogenous equation $\Delta u = 0$ in Ω with an arbitary continuous boundary data.

We now give a useful charaterization of the regularity for the Dirichlet problem.

Theorem 1.36. *Let $\Omega \Subset \mathbb{C}$ be a domain with non empty boundary. Then the following properties are quivalent:*

(i) Ω is regular for the (classical) Dirichlet problem,

(ii) the Dirichlet problem $\mathrm{DP}(\Omega, \phi, \{\mathbf{0}\})$ *is solvable for any continuous boundary data ϕ.*

(iii) there exists a continuous subharmonic function ρ in Ω such that $\Delta\rho \geq 1$ weakly on Ω and for any $\zeta \in \partial\Omega$

$$\lim_{z \to \zeta} \rho(z) = 0.$$

Such function is called a bounded strictly subharmonic exhaution for Ω.

Proof. The implication $(i) \implies (ii)$ follows from Theorem 1.34. To prove the implication $(ii) \implies (iii)$, consider the solution h of the homogenous Dirichlet problem with boundary value $\phi(\zeta) := -|\zeta|^2/2$ for $\zeta \in \partial\Omega$. Then set

$$\rho(z) := h + |z|^2/2.$$

is bounded stricly subharmonic exhaution for Ω.

Now let us prove that $(iii) \Longrightarrow (i)$. Fix $\zeta \in \partial\Omega$ and consider the subharmonic exhaution ρ so that $\Delta\rho \geq 1$ in Ω. Then the function defined by $b_\zeta(z) := 2\rho(z) - |z-\zeta|^2$ is a continuous function in $\bar{\Omega}$ such that $\Delta b_\zeta \geq 1$ on Ω, hence b_ζ is subharmonic in Ω and provides a subbarrier at the point ζ. □

It is easy to deduce from this result the existence of the Green kernel.

COROLLARY 1.37. *Let Ω be a regular domain for the Dirichlet problem. Then Ω admits a Green function with logarithmic pole at any fixed point $\xi \in \Omega$.;*

PROOF. Indeed, fix a point $\xi \in \Omega$ and solve the Dirichlet problem for the homogenuous Laplace equation in Ω with boundary data $h(\zeta) := -\log|\zeta - \xi|$, which is a continuous function on $\partial\Omega$. If u is the solution, then the function defined by $g(z) := u(z) + \log|z-\xi|$ is subharmonic in Ω, harmonic in $\Omega \setminus \{\xi\}$ and has a logarithmic pole at the point ξ. Since it has zero boundary values, it coincides with the Green function $G_\Omega(\cdot,\xi)$. □

We now give some examples and applications of the previous theorem.

EXAMPLE 1.38. *1. Let $\mathbb{D}$ be the unit idsc. Then for any fixed $w \in \mathbb{D}$, the Grenn function $G_{\mathbb{D}}(\cdot, w)$ of $\mathbb{D}$ with logarithmic pole at w is given by the formula*

$$G_{\mathbb{D}}(z,w) = \log\frac{|z-w|}{|1-\bar{w}\cdot z|},\ z \in \mathbb{D}.$$

2. Let $F : \Omega \longrightarrow \mathbb{D}$ be an biholomorphic function. Then for any fixed $\zeta \in \Omega$, the Grenn function $G_\Omega(\cdot,\zeta)$ of Ω with logarithmic pole at ζ is given by the formula

$$G_\Omega(z,\zeta) = \log\frac{|F(z)-F(\zeta)|}{|1-\bar{F}(\zeta)\cdot F(z)|},\ z \in \Omega.$$

In particular, if $F(\zeta) = 0$ then $G_\Omega(z,\zeta) = \log|F(z)|$ for $z \in \Omega$.

COROLLARY 1.39. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain such that $\Omega \neq \mathbb{C}$. Then we have the following properties:*

(I) for any fixed point $\zeta \in \Omega$, there exists a Green function $G_\Omega(\cdot,\zeta)$ with logarithmic pole at ζ;

(ii) the function $G_\Omega(\cdot,\zeta)$ is the upper envelope of all $u \in SH(\Omega)$ such that $u \leq 0$ in Ω and $u(z) - \log|z-\zeta|$ is upper bounded near w.

(iii) the Green function is symmetric i.e. $\forall z, \zeta \in \Omega$, $G_\Omega(z,\zeta) = G_\Omega(\zeta, z$.

Proof. We first reduce to the case when $\Omega \Subset \mathbb{D}$ is bounded. Indeed, we will show that Ω is isomorphic to a bounded simply connected domain.

Since Ω is a proper simply connected subdomain of $\mathbb{C}$, its boundary contains at least two distinct points. Composing with a Möbius transform we can assume that these two points are 0 and 1. Then since Ω is simply connected, it follows that the square root function defined by

$$f(z) := \sqrt{\frac{z}{z-1}},$$

is holomorphic in Ω. Actually the square root function has two branches $f_1 = f$ and $f_2 = -f$. Since $f_1^2 = f_2^2$ is injective, it follows that f_1 and f_2 are injective holomorphic open functions on Ω. Moreover $f_1(\Omega) \cap f_2(\Omega) = \emptyset$ since $f(z) \neq 0$ in Ω. Therefore $f_2(\Omega) \subset \mathbb{C} \setminus f_1(\Omega)$ and there exists a disc $\bar{\mathbb{D}}(a,r) \subset \mathbb{C} \setminus f_1(\Omega)$. Then the function defined by

$$h(z) := \frac{r}{f_1(z) - a}, \ z \in \Omega,$$

is holomorphic and injective on Ω and its image is contained in the unit disc. Hence it is an isomorphism from Ω onto a bounded simply connected domain.

1. We show that the domain Ω is regular for the Dirichlet problem. Indeed fix $\zeta \in \partial\Omega$. Then by translation and dilation, we can assume that $\zeta = 0 \in \partial\Omega$. Since Ω is a simply connected domain in $\mathbb{C} \setminus \{0\}$, there exists a holmorphic branch log of the logarithm on Ω. Therefore the function defined on Ω by

$$b_0(z) := \Re(1/\log z)$$

is a subharmonic barrier function for Ω at the boundary point $\zeta = 0$.

2. Let u be as in the statement and set

$$v(z) := u(z) - G_\Omega(z,\zeta), z \in \Omega.$$

The function u is subharmonic in $\Omega \setminus \{\zeta\}$ and upper bounded near ζ. Then it extends to a subharmonic function in Ω such that $\limsup_{z\to\xi} v(z) \leq 0$ for any $\xi \in \partial\Omega$. By the classical maximum principle, we get $v \leq 0$ in Ω, hence $u \leq G_\Omega(z,\zeta)$.

3. The symmetry of the Green kernel follows from (6.1) in step 2 of the proof of the Riemann mapping theorem given below. □

Remark 1.40. *The Green function with logarithmic pole at some fixed point* $\xi \in \Omega$ *gives a weak subbarrier at the boundary point* $\xi \in \partial$. *One can prove that a (strong) subbarrier at such point of the boundary exists. This is the content of Bouligand's Lemma (see* [**Ra95**, Lemma

4.1.7]*). Hence for a domain $\Omega \Subset \mathbb{C}$, the existence of the Green kernel G_Ω is equivalent to the regularity for the Dirichlet problem.*

5.3. The non homogeneous Dirichlet problem. Here we consider the Dirichlet problem for the Laplace equation in a bounded domain $\Omega \Subset \mathbb{C}$ with zero boundary value and right hand side μ a positive Borel measure on Ω. We will assume that Ω is regular for the dirichlet problem. Then by Theorem 1.36 Ω has a bounded stricly subharmonic exhaution functin ρ such that $\Delta \rho \geq 2$ in the weak sense on Ω.

We first observe that in general there is no solution.

EXAMPLE 1.41. *Let $\Omega = \mathbb{D}$ be the unit disc. We know by Poisson-Jensen formula 1.31 that the solution V of $\mathrm{DP}(\Omega, \phi, \{\mathbf{0}\})$, if it exists, is given as the Green potential of the measure mu:*

$$V(z) := \frac{1}{2\pi} \int_{\mathbb{D}} G_{\mathbb{D}}(z, \zeta) d\mu(\zeta),$$

since $V = 0$ in $\partial \mathbb{D}$. Let μ be a discrete measure given by $\mu = \sum_{j \in \mathbb{N}} 2^{-j} \delta_{a_j}$, where $(a_j)_{j \in \mathbb{N}}$ is a sequence of points in $\mathbb{D}$ converging to 1. Assume that the Dirichlet problem $\mathrm{DP}(\Omega, 0, \mu)$ has a solution. Then it is given by

$$V(z) = \sum_{j \in \mathbb{N}} \varepsilon_j G_{\mathbb{D}}(z, a_j).$$

However since $V(a_j) = -\infty$, we have $\liminf_{z \to 1} V(z) = -\infty$, and V has not zero boundary values, which is a contradiction.

Hence the the non homogenuous Dirichlet problem in $\mathbb{D}$ with boundary values 0 and right hand side μ has no solution.

Observe that $\limsup_{z \to \zeta} V(z) = 0$ for any $\zeta \in \partial \mathbb{D}$. The problem is that there is no subbarrier for the problem at the boundary point 1.

We can prove the following result.

THEOREM 1.42. *Assume that Ω is regular for the classical Dirichlet problem and let μ a Borel measure on Ω with density $f \geq 0$ continuous function in $\bar{\Omega}$. Then the Dirichlet problem with boundary values 0 and right hand side f has a unique solution. It is the maximal subsolution to the problem.*

PROOF. Let ρ be a defining subharmonic function for Ω such that $\Delta \rho \geq 1$. Since $f \leq M := \max_{\bar{\Omega}} f$, it follows that the function $v(z) := A\rho(z)$ is subharmonic in Ω and for $\Delta v = A$. Then if $A > M$, it follows that v is a continuous subbarrier for the Dirichlet problem

Hence the upper envelope of subsolution V_f is a negative subhamonic function in Ω such that $v \leq V_\mu \leq 0$. Since v has zero at the boundary, it follows that V_f also has zero boundary values.

To prove that $\Delta V_f = f$ in the sense of distributions, we cannot apply the balayage procedure since we have to solve a more complicated problem. We must proceed differently.

Assume fist that f is smooth with compact support in Ω. Then consider the logarithmic potential of the measure with density f defined as follows:

$$v(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \log |z - \zeta| f(\zeta) d\lambda(\zeta).$$

Then v is subharmonic in $\mathbb{C}$, harmonic in a neighborhood of the boundary (in fact in the complement of the support of f) and $\Delta v = f$ weakly on Ω. Since v is continuous in $\partial\Omega$, by Theorem 1.34, there exists a harmonic function h in Ω wich coincides with $-v$ at the boundary. Hence $v + h$ is the solution of the Dirichlet problem $\mathrm{DP}(\Omega, 0, \mathrm{f})$. By the maximum principle it coincides with the upper envelope V_f.

In the general case, we approximate f uniformly in $\bar{\Omega}$ by smooth functions (f_j) with compact support in Ω. Denote by V_j the solution for the Dirichlet problem $\mathrm{DP}(\Omega, 0, \mathrm{f_j})$. We claim that the sequence (V_j) converges uniformly to $V := V_f$.

We need a stability property. We claim that there exists a constant $C > 0$ such that for any $j \in \mathbb{N}$,

$$\|V_j - V\|_{L^\infty(\Omega)} \leq C \|f_j - f\|_{L^\infty(\Omega)}$$

By Theorem 1.34, there exists a harmonic function h in Ω such that $h = -|z|^2/2$ in $\partial\Omega$. Then the function $\psi(z) := h(z) + |z|^2$ solves tha Dirichlet problem for $\Delta\psi = 1$ with boundary value 0.

Set $\varepsilon_j := \|f_j - f\|_{L^\infty(\Omega)}$. Then by the maximum principle, the inequality $f \leq f_j + \varepsilon_j$ implies that

$$V_j + \varepsilon_j \psi \leq V,$$

while the inequality $f_j \leq f + \varepsilon_j$ implies the inequality

$$V + \varepsilon_j \psi \leq V_j.$$

The two inequalities imply that $|V_j - V| \leq -\varepsilon_j \psi$ in Ω, which proves our claim with the constant $C := \max_{\bar{\Omega}}(-\psi)$.

We have proved that $V_j \to V$ uniformly in Ω. Then it follows that $\Delta V_j \to \Delta V$ in the weak sense on Ω, hence $\Delta V = f$ in the weak on Ω, which proves our theorem. □

In the general case, we can prove the following result.

THEOREM 1.43. *Let Ω be a bounded domain regular for the Dirichlet problem and let μ a Borel measure with finite mass.*

Then we have the following properties:

1. if μ has compact support, then its Green potential V_μ given bu the formula

$$V_\mu(z) := \int_\Omega G_\Omega(z, \zeta) d\mu(\zeta),$$

is the unique solution to the Dirichlet problem $\mathrm{DP}(\Omega, 0, \mu)$.

2. if there exists $v \in SH(\Omega)$ such that

$$\Delta v \geq \mu \text{ on } \Omega, \text{ and } v = 0 \text{ in } \partial\Omega,$$

the Green potential V_μ *is a solution to the Dirichlet problem* $\mathrm{DP}(\Omega, 0, \mu)$.

The theorem can be stated quickly by saying that if the problem has a subsolution (meaning a subbarier) then it has a solution.

PROOF. Assume first that μ has compact support. Then V_μ is subharmonic in Ω and harmonic in $\Omega \setminus \mathrm{Supp}\mu$. Let $D \Subset \Omega$ be a domain such that $\mathrm{Supp}\mu \subset D \Subset \Omega$. We claim that there exists a constant m such that for any $z \in \Omega \setminus \mathrm{Supp}$ and $\zeta \in \mathrm{Supp}$, we have

$$m \leq G_\Omega(z, \zeta) \leq 0.$$

Assuming the claim, it follows from Lebesgue convergence theorem that $\lim_{z \to \zeta} V_\mu(z) = 0$. To prove the claim, observe that when $\Omega = D$ is a disc, from the explicit formula of G_D we see that it is continuous in $\bar{D} \times \bar{D} \setminus \Sigma$, where Σ is the diagonal in $\mathbb{C}^2$. In the general case, take a big disc $D = D(0, R)$ containing Ω and observe that by the comparaison principle for any $\zeta \in \Omega$, we have $G_D(z, \zeta) \leq G_\Omega(z, \zeta) \leq 0$.

Now when the support of μ is not compact, we can consider an increasing sequence (K_j) of compact sets such that $\Omega = \cup_j K_j$. Let $\mu_j := \mathbf{1}_{K_j} \mu$ the restriction of the Borel measure μ to K_j. Then by the first part the corresponding Green potential $V_j := V_{\mu_j}$ are subharmonic, zero boundary values and $\Delta V_j = \mu_j$ on Ω. By the comparison principle since $\mu_j \leq \mu_{j+1} \leq \mu \leq (dd^c v)^n$ weakly on Ω, it follows that $v \leq V_{j+1} \leq V_j \leq 0$ in Ω.

By the monotone convergence, it follows that the sequence (V_j) converge to V in Ω and $v \leq V \leq 0$ in Ω, which implies that V has zero boundary values and then is the solution to $\mathrm{DP}(\Omega, 0, \mu)$. □

REMARK 1.44. *In the case when* μ *has a density such that* $0 \leq f \in L^p(\Omega)$ *with* $p > 1$, *it is possible to show the existence of a solution which is continuous up the boundary. Here the stability inequality is much more complicated to prove.*

6. Application : The Riemann mapping theorem

As an application of the previous results, we give an elegant potential proof of a fundamental theorem in Complex Analysis, called the *Riemann mapping theorem.*

THEOREM 1.45. *Let* $\Omega \subset \mathbb{C}$ *be a simply connected domain such that* $\Omega \neq \mathbb{C}$. *Then there exists a unique holomorphic isomorphism* $F = F_w$ *of* Ω *onto the unit disc* $\mathbb{D}$ *such that* $F(w) = 0$ *and* $F'(w) > 0$ *and*

$$|F_w(z)| := e^{G_\Omega(z,w)}, \ z \in \Omega.$$

PROOF. **Step 1 :** *Construction of a proper holomorphic map* from Ω onto $\mathbb{D}$. By the Theorem 1.39, Ω admits a Green function G_Ω with pole at any $w \in \Omega$. Let $h(z) := G_\Omega(z, w) - \log|z - w|$ for $z \in \Omega \setminus \{w\}$. Then the function h is harmonic in $\Omega \setminus \{w\}$ and locally bounded near

w in $\Omega \setminus \{w\}$. Therefore it extends into a subharmonic function in Ω by Proposition 1.17. Let h^* be the unique harmonic conjugate function of h in Ω such that $h^*(w) = 0$. Then the function defined on Ω by

$$F_w(z) := (z - w)e^{h+ih^*}$$

is a holomorphic function in Ω such that $|F_w(z)| = e^{G_\Omega(z,w)} < 1$ for any $z \in \Omega$ and $\lim_{z \to \partial\Omega} |F_w(z)| = 1$. Therefore F_w is a proper holomorphic function from Ω onto $\mathbb{D}$ which vanishes only at w.

Step 2 : *The Green function is symmetric.* Let $w, \zeta \in \Omega$ be fixed. Let $\xi := F_w(\zeta)$ and $\phi \in Aut(\mathbb{D})$ be defined by

$$\phi(z) := \frac{z - \xi}{1 - \bar{\xi} z}, z \in \mathbb{D}.$$

Then $\phi \circ F_w(\zeta) = 0$ and $\phi(0) = -\xi$. Therefore the function $\phi \circ F_w$ is holomorphic in Ω with value in $\mathbb{D}$ and vanishes at ζ. Hence $u := \log |\phi \circ F_w|$ is subharmonic, ≤ 0 and $u(z) - \log |z - \zeta$ is upper bounded near ζ. Hence by the previous result we obtain $u \leq G_\Omega(\cdot, \zeta)$.

Therefore $|\phi \circ F_w| \leq |F_\zeta|$ in Ω. Applying this inequality at the point $z = w$ gives $|\phi \circ F_w(w)| \leq |F_\zeta(w)|$.

Observing that $\phi \circ F_w(w) = \phi(0) = -F_w(\zeta)$, we obtain the inequality $|F_w(\zeta)| \leq |F_\zeta(w)|$. Reversing the role of : $zeta$ and w we obtain the equality $|F_w(\zeta)| = |F_\zeta(w)|$, which means exactely the symmetry of the Green function:

$$G_\Omega(\zeta, w) = G_\Omega(w, \zeta), \forall (z, \zeta) \in \Omega \times \Omega. \tag{6.1}$$

Step 3 : *F_w is injective in Ω.* Take another point $\zeta \in \Omega$, $\zeta \neq w$ and consider the function

$$u(z) := G_\mathbb{D}(F_w(z), F_w(\zeta)) - G_\Omega(z, \zeta).$$

Then u is subharmonic in $\Omega \setminus \{\zeta\}$ and bounded from above near ζ. Therefore it extends as a subharmonic function in Ω, denoted by u. Since u tends to 0 at the boundary of Ω, it follows from the maximum principle that $u \leq 0$ in Ω. Since $|F_w(\zeta)| = |F_\zeta(w)|$ by step 2, it follows that

$$u(w) = G_\mathbb{D}(0, F_w(\zeta)) - G_\Omega(w, \zeta) = \log |F_w(\zeta)| - |F_\zeta(w)| = 0,$$

Therefore $u \equiv 0$ in Ω by the maximum principle. This means that

$$G_\mathbb{D}(F_w(z), F_w(\zeta)) = G_\Omega(z, \zeta),$$

which implies that F_w is injective. Indeed, if $z, w \in \Omega$ and $z \neq w$ then $G_\Omega(z, \zeta) > -\infty$, hence $G_\mathbb{D}(F_w(z), F_w(\zeta) > -\infty$ and then $F_w(z) \neq F_w(\zeta$. $\square$

The theorem may have been suggested to Riemann by physical considerations of fluid flow or electric fields in such domains, for he made use of the Dirichlet principle which was proved later by D. Hilbert.

As a consequence, it is useful to see this theorem as the classification of simply connectd domains in the Riemann sphere.

We consider the compactification of $\mathbb{C}$ by adding an extra point at infinity denoted ∞ i.e.

$$\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

We identify $\mathbb{C} = \{z = x + iy, (x, y) \in \mathbb{R}^2\}$ to the equatorial plan in $\mathbb{R}^3$ given by $\{(\xi, \eta, \zeta) \in \mathbb{R}^3; \zeta = 0\}$. Then it is easy to show that $\bar{\mathbb{C}}$ is homeomorphic to the unit sphere

$$\mathbb{S}^2 := \{(\xi, \eta, \zeta) \in \mathbb{R}^3; \xi^2 + \eta^2 + \zeta^2 = 1\}.$$

The homeomorphic is defined by the stereographic projection from the north pole: let $N := (0, 0, 1)$ be the north pole of the unit sphere. Given a point $P = (\xi, \eta, \zeta)\mathbb{S}^2 \setminus \{N\}$, we associate to P the unique point of intersection of the real line NP with the equatorial place $\mathbb{C} \subset \mathbb{R}^3$. We denote this point by $M = \pi_N(P) \in \mathbb{C}$. The map $\pi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is an homeomorphism. An easy computation show that if $P = (\xi, \eta, \zeta)\mathbb{S}^2 \setminus \{N\}$ then its image $z = \pi_N(P)$ is given by

$$z = \frac{\xi + i\eta}{1 - \zeta}.$$

It is clear that π_N extends to an an homemorphism $\pi_N : \mathbb{S}^2 \simeq \bar{\mathbb{C}}$ by setting $\pi_N(N) = \infty$.

We could also consider the stereographic projection from the south pole $S := (0, 0, -1)$. Then map $\pi_S : \mathbb{S}^2 \setminus \{S\} \to \mathbb{C}$ is an homeomorphism. An easy computation show that if $P = (\xi, \eta, \zeta) \in \mathbb{S}^2 \setminus \{S\}$ then its image $w = \pi_S(P)$ is given by

$$w = \frac{\xi + i\eta}{1 + \zeta}.$$

Observe that for any $P = (\xi, \eta, \zeta) \in \mathbb{S}^2 \setminus \{N, S\}$ we have $z\bar{w} = 1$. In other words the map $\pi_S \circ \pi_N^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ sends $z \in \mathbb{C}$ to $w = 1/\bar{z}$ is an inversion with pole at the origin.

In modern language we say that $\mathcal{S}^2$ carries a structure of a Riemann surface. This means that $\mathcal{S}^2 = \mathcal{U} \cup \mathcal{V}$ is covered by two open sets with the following homeomorphisms:

The stereographic projection from the north pole N

$$\pi_N : \mathcal{U} := \mathbb{S}^2 \setminus \{N\} \longrightarrow \mathbb{C}$$

and the conjugate stereographic projection from the south pole S

$$\pi_S^* : \mathcal{V} := \mathbb{S}^2 \setminus \{S\} \longrightarrow \mathbb{C}$$

which satifies the holomorphic compatibility condition that

$$\pi_S^* \circ \pi_N^{-1} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$$

is the holomorphic homeomorphism $z \longmapsto 1/z$.

The two coordinates charts $(\mathcal{U}, \pi_N)$ and $(\mathcal{U}, \pi_S^*)$ are called a holomorphic atlas and allows to define the notion of holomorphic function on an open set of the Riemann sphere. A function defined in a neiborhood of ∞ is holomorphic if the function $z \longmapsto f(1/z)$ is holomorphic in a neighborhood of the origin.

Then we can obtain the following classification theorem.

COROLLARY 1.46. *Let Ω be a simply connected domain in the Riemann sphere $\bar{\mathbb{C}}$. Then we have the following trichotomy: either $\Omega = \bar{\mathbb{C}}$, or $\Omega = \mathbb{C}$, or Ω is isomorphic to one the unit disc $\mathbb{D}$.*

This statement is is known as the "Uniformization Theorem" of Riemann and this trichotomy corresponds to the three possible geometries that exist on a Riemann surface.

The general version of the Uniformization Theorem can be stated as follows:

Uniformisation Theorem. *Every simply connected Riemann surface is conformally equivalent to the unit disk, the complex plane, or the Riemann sphere.*

The uniformization theorem was first proved by Koebe and Poincaré independently in 1907. It is a classification theorem of all Riemann surfaces according to their universal covering spaces into three groups. Importantly, it reduces many aspects of Riemann surfaces to the study of the disk, the plane, and the Riemann sphere.

7. Historical comments

As we already said, the term potential theory arises from the fact that, in 19^{th} century physics, the fundamental forces of nature were believed to be derived from potentials which satisfied Laplace's equation. Hence, potential theory was the study of functions that could serve as potentials. Nowadays, we know that nature is more complicated - the equations that describe forces are systems of non-linear partial differential equations, such as the Einstein equations, the Yang-Mills equations, and that the Laplace equation is only valid as a limiting case. Nevertheless, in Mathematical Physics, the term potential theory has remained as a convenient term for describing the study of functions satisfying the Laplace equation and its generalisations.

As it was mentioned, originally studies were related to the properties of forces which follow the law of gravitation. In the statement of this law given by I. Newton $(1643 - 1727)$ in 1687 the only forces considered were the forces of mutual attraction acting upon two material particles of small size or two material points. These forces are directly proportional to the product of the masses of these particles and inversely proportional to the square of the distance between them. Thus, the first and the most important problem from the point of view of celestial mechanics and geodesy was to study the forces of attraction of a material point by a finite smooth material body - a spheroid and, in particular, en ellipsoid (since many celestial bodies have this shape). After first partial achievements by Newton and others, studies carried out by J. L. Lagrange $(1736-1813)$ in 1773, A. Legendre $(1752-1833)$ between $1784-1794$ and by P. S. Laplace $(1749-1827)$ continued in $1782-1799$ became of major importance. Lagrange established that the field of gravitational forces, as it is called now, is potential field. He introduced a function which was called in 1828 by G. Green $(1793-1841)$ a potential function and later in 1840 by C. F. Gauss $(1777-1855)$ - just a potential. At present, the achievements of this initial period are included in courses on celestial mechanics.

Already Gauss and his contemporaries discovered that the method of potentials can be applied not only to solve problems in the theory of gravitation but, in general, to solve a wide range of problems in mathematical physics, in particular, in electrostatics and magnetism. In this connection, potential became to be considered not only for the physically realistic problems concerning mutual attraction of positive masses, but also for problems with masses of arbitrary sign, or charges. Important boundary value problems were defined, such as the Dirichlet problem and the Neumann problem, the electrostatic problem of the static distribution of charges on conductors or the Robin problem, and the problem of sweeping-out mass (balayage method). To solve the abovementioned problems in the case of domains with sufficiently smooth boundaries certain types of potentials turned out to be efficient, i. e., special classes of parameter - dependent integrals such as volume potentials of distributed mass, single - and double - layer potentials, logarithmic potentials, Green potentials, etc. Results obtained by A. M Lyapunov $(1857-1918)$ and V. A. Steklov $(1864-1926)$ at the end of 19^{th} century were fundamental for the creation of strong methods of solution of the main boundary value problems. Studies in the potential theory concerning properties of different potentials have acquired an independent significance.

In the first half of the 20^{th} century, a great stimulus for the generalisation of the principal problems and the completion of the existing formulations in the potential theory was made on the basis of the general notions of a Radon measure, capacity and generalised functions. Modern potential theory is closely related in its development to the theory of analytic, harmonic and subharmonic functions and to the probability theory. Together with further studies of classical boundary value problems and inverse problems, the modern period of the development of potential theory is characterised by the applications of methods and notions of topology and functional analysis, and the use of abstract axiomatic methods.

8. Exercises

Exercise 1.1.

1) Show that a (real valued) function u (on a metric space) is upper semi-continuous iff $\limsup_{z\to a} u(z) = u(a)$ *at all points a.*

2) Let $u : X \to \mathbb{R} \cup \{-\infty\}$ *be a bounded upper semi-continuous function on a metric space* (X, d). *Show that*

$$x \mapsto u_k(x) := \sup\{u(y) - kd(x, y); y \in X\}$$

are Lipschitz functions which decrease to u as k increases to $+\infty$,

3) Same question if u is merely bounded from above (replace u by $(\sup\{u, -j\})_{j\in\mathbb{N}}$ *and use previous question).*

Exercise 1.2. *Let* $u_1, \dots, u_s$ *be subharmonic functions in a domain* $\Omega \subset \mathbb{C}$. *Show that*

$$v := \log\left[e^{u_1} + \cdots + e^{u_s}\right]$$

defines a subharmonic function in Ω.

Deduce from this by a rescaling argument that $\max(u_1, \dots, u_s)$ *is subharmonic as well.*

Exercise 1.3. *Let I be an open subset of* $\mathbb{R}$. *Show that* $f : I \to \mathbb{R}$ *is convex if and only if*

$$\limsup_{h\to 0}\left[\frac{f(x+h) + f(x-h) - 2f(x)}{h^2}\right] \geq 0.$$

Exercise 1.4. *Let I be an open subset of* $\mathbb{R}$ *and* $f : I \to \mathbb{R}^+_*$ *a positive function. Show that* $\log f : I \to \mathbb{R}$ *is convex if and only if for all* $c \in \mathbb{R}$, $t \in I \mapsto e^{ct} f(t) \in \mathbb{R}$ *is convex.*

Exercise 1.5. *Let* $f_j : \mathbb{R} \to \mathbb{R}$ *be a sequence of convex functions which converge pointwise towards a function* $f : \mathbb{R} \to \mathbb{R}$. *Show that f is convex and that* (f_j) *uniformly converges towards f on each compact subset of* $\mathbb{R}$.

Exercise 1.6. *Let* $\Omega \subset \mathbb{C}$ *be an open subset of* $\mathbb{C}$ *and* $u : \Omega \to \mathbb{R}^+_*$ *a positive function. Show that* $\log u : \Omega \to \mathbb{R}$ *is subharmonic in* Ω *if and only if for all* $a \in \mathbb{C}$, *the function* $v_a : z \in \Omega \mapsto e^{\Re(az)} u(z)$ *is subharmonic in* Ω. *(Consider first the case when u is smooth).*

Exercise 1.7.

1. Compute the Laplacian in polar coordinates in $\mathbb{C}$.

2. Let $u(z) = \chi(|z|)$, χ *a smooth function in* $[0, R[$. *Show that*

$$\Delta u(z) = \chi''(r) + \frac{1}{r}\chi'(r).$$

3. Describe all harmonic radial functions in $\mathbb{C}$.

4. Show that u is subharmonic in a disc $\mathbb{D}(0, R)$ *iff* χ *is a convex increasing function of* $t = \log r$ *in the interval* $]-\infty, \log R[$.

EXERCISE 1.8. *Let* $h : \mathbb{R}^2 \to \mathbb{R}$ *be a harmonic function. Assume there exists* $C, d > 0$ *such that*

$$|h(x)| \leq C[1 + ||x||]^d, \quad \text{for all } x \in \mathbb{R}^2.$$

Show that h *is a polynomial of degree at most* d.

EXERCISE 1.9. *Let* $\phi : \partial\mathbb{D} \to \mathbb{R}$ *be a continuous function and let* u_ϕ *denote the Poisson transform of* ϕ *in the unit disc* $\mathbb{D} \subset \mathbb{C}$.

i) Show that u_ϕ *is Hölder continuous on* $\overline{\mathbb{D}}$ *if and only if* ϕ *is Hölder continuous. Is the Hölder exponent preserved ?*

ii) By considering $\phi(e^{i\theta}) = |\sin\theta|$, *show that* ϕ *Lipschitz on* $\partial\mathbb{D}$ *does not necessarily imply that* u_ϕ *is Lipschitz on* $\overline{\mathbb{D}}$.

EXERCISE 1.10. *Let* μ *be a probability measure in* $\mathbb{C}$.

1) Show that

$$\varphi_\mu(z) := \int_{w\in\mathbb{C}} \log|z - w| d\mu(w)$$

defines a subharmonic function with logarithmic growth in $\mathbb{C}$.

2) Show that if φ *is a subharmonic function with logarithmic growth in* $\mathbb{C}$, *then there exists* $c \in \mathbb{R}$ *such that* $\varphi = \varphi_\mu + c$, *where* $\mu = \Delta\varphi/2\pi$.

3) Approximating μ *by Dirac masses, show that every subharmonic function with logarithmic growth in* $\mathbb{C}$ *can be approximated in* L^1 *by functions of the type* $j^{-1}\log|P_j|$, *where* P_j *is a polynomial of degree* j.

EXERCISE 1.11. *Let* $(a_j) \in \mathbb{C}^{\mathbb{N}}$ *be a bounded sequence which is dense in the unit disc and let* $\varepsilon_j > 0$ *be positive reals such that* $\sum_j \varepsilon_j < +\infty$. *Show that the function*

$$z \mapsto u(z) := \sum_j \varepsilon_j \log|z - a_j|$$

belongs to $SH(\mathbb{C})$ *and has an uncoutable polar set* $(u = -\infty)$.

Check that u *is discontinuous almost everywhere in the unit disc. (See* [**Ra95**]*)*

EXERCISE 1.12. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and* $T \in \mathcal{D}'(\Omega)$ *be a non-negative distribution, i.e.*

$$\langle T, \chi \rangle \geq 0$$

for all non-negative smooth test functions $0 \leq \chi \in \mathcal{D}(\Omega)$. *Show that* T *is of order zero, i.e. it can be extended as a continuous (non-negative) linear form on the space of continuous functions with compact support in* Ω *(in other words* T *extends as a Radon measure).*

CHAPTER 2

Plurisubharmonic functions

As we have seen in the previous chapter, subharmonic functions are intimately related to holomorphic functions and Logarithmic potential Theory has interesting applications to Complex Analysis in one variable.

Plurisubharmonic functions have been introduced independently in 1942 by Pierre Lelong in France and Kiyoshi Oka in Japan as generalizations to higher dimension of subharmonic functions .

Oka used them to define pseudoconvex domains and solved the Levi problem in dimension two. Lelong established their first properties and was the first to ask for a pluricomplex counterpart of Potential Theory in $\mathbb{C}^n$, especially after the important work of Cartan on Newtonian Potentials ([**Car45**]). He posed influential problems, some of which named the first and the second Lelong problem remained open for decades.

These problems have been eventually solved by Bedford and Taylor in two landmark papers [**BT76, BT82**] which laid down the foundations of the now called *Pluripotential Theory*.

Our purpose in this notes is to develop the first steps of Bedford-Taylor Theory. We haven't tried to make an exhaustive presentation, we merely present those results that will be used in the sequel, when adapting this theory to the setting of compact Kähler manifolds.

On the other hand, plurisubharmonic functions are in many ways analogous to convex functions. They relate to subharmonic functions of one complex variable as convex functions of several variables do to convex functions of one real variable. On the other hand plurisubharmonic functions can have singularities (they are not necessarily continuous, nor even locally bounded). This makes questions of local regularity much trickier than for convex functions.

One needs infinitely many (sub mean-value) inequalities to define plurisubharmonic functions, this is best expressed in the sense of currents (differential forms with coefficients distributions): a function φ is plurisubharmonic if and only if the current $dd^c\varphi = \sqrt{-1}\partial\bar{\partial}\varphi$ is a positive current (see next chapter).

In this chapter we establish basic properties of plurisubharmonic functions. We first extend the basic properties of subharmonic functions in the plane to the pluricomplex case, and then move on to study plurisubharmonic functions (defined as upper semi-continuous functions whose restriction to any complex line is subharmonic).

We give several examples and establish important compactness and integrability properties of families of plurisubharmonic functions.

We encourage the readers who wish to learn more about pluripotential theory in domains of $\mathbb{C}^n$ to consult the excellent surveys that are available, notably [**Sad81, Bed93, Ceg88, Dem91, Kis00, Klim91, Blo02, Kol05**].

1. Plurisubharmonic functions

We now introduce the fundamental objects that we are going to study in the sequel. The notion of plurisubharmonic function is the pluricomplex counterpart of the notion of subharmonic function.

1.1. Basic properties. We fix Ω a domain of $\mathbb{C}^n$.

DEFINITION 2.1. *A function $u : \Omega \longrightarrow [-\infty + \infty[$ is plurisubharmonic if it is upper semi-continuous and for all complex lines $\Lambda \subset \mathbb{C}^n$, the restriction $u|\Omega \cap \Lambda$ is subharmonic in $\Omega \cap \Lambda$.*

The latter property can be reformulated as follows: for all $a \in \Omega$, $\xi \in \mathbb{C}^n$ with $|\xi| = 1$ and $r > 0$ such that $\bar{B}(a, r) \subset \Omega$,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}\xi) d\theta. \tag{1.1}$$

All basic results that we have established for subharmonic functions are also valid for plurisubharmonic functions. We state them and leave the proofs to the reader:

PROPOSITION 2.2.

(1) *If $u : \Omega \longrightarrow [-\infty, +\infty[$ is plurisubharmonic in Ω and χ is a real convex increasing function on an interval containing the image $u(\Omega)$ of u then $\chi \circ u$ is plurisubharmonic in Ω.*
(2) *Let $(u_j)_{j \in \mathbb{N}}$ be a decreasing sequence of plurisubharmonic functions in Ω. Then $u := \lim_{j \to +\infty} u_j$ is plurisubharmonic in Ω.*
(3) *Let $(X, \mathcal{T})$ be a measurable space, μ a positive measure on $(X, \mathcal{T})$ and $E(z, x) : \Omega \times X \longrightarrow \mathbb{R} \cup \{-\infty\}$ be such that*
(i) for μ-a.e. $x \in X$, $z \longmapsto E(z, x)$ is plurisubharmonic,
(ii) $\forall z_0 \in \Omega$ there exists $r > 0$ and $g \in L^1(\mu)$ such that for all $z \in B(z_0, r)$ and μ-a.e. $x \in X$, $E(z, x) \leq g(x)$.
Then $z \mapsto V(z) := \int_X E(z, t) d\mu(x)$ is plurisubharmonic.

Recall that a function $f : z = (z_1, \ldots, z_n) \in \Omega \mapsto f(z) \in \mathbb{C}$ is holomorphic if it satisfies the Cauchy-Riemann equations $\partial f / \partial \overline{z}_j = 0$ for all $1 \leq j \leq n$.

PROPOSITION 2.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$ and f a holomorphic function such that $f \not\equiv 0$ in Ω. Then $\log |f|$ is plurisubharmonic* *in* Ω *and pluriharmonic in* $\Omega \setminus \{f = 0\}$. *Moreover for any* $\alpha > 0$, $|f|^\alpha$ *is plurisubharmonic in* Ω.

A function is pluriharmonic if it satifies the linear equations

$$\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} = 0$$

for all $1 \leq j, k \leq n$. One shows (like in complex dimension 1) that a function is pluriharmonic if and only if it is locally the real part of a holomorphic function.

We add one more recipe known as the *gluing construction*:

Proposition 2.4. *Let* u *be a plurisubharmonic function in a domain* Ω. *Let* v *be a plurisubharmonic function in a relatively compact subdomain* $\Omega' \subset \Omega$. *If* $u \geq v$ *on* $\partial\Omega'$, *then the function*

$$z \longmapsto w(z) = \begin{cases} \max[u(z), v(z)] & \text{if } z \in \Omega' \\ u(z) & \text{if } z \in \Omega \setminus \Omega' \end{cases}$$

is plurisubharmonic in Ω.

Proof. The upper semi-continuity property is clear. Replacing v by $v - \varepsilon$, one gets that u strictly dominates $v - \varepsilon$ in a neighborhood of $\partial\Omega'$ and the corresponding function w_ε is then clearly plurisubharmonic. Now w is the increasing limit of w_ε as ε decreases to zero, so it satisfies the appropriate submean-value inequalities. □

1.2. Submean-value inequalities. The following result follows from its analogue in one complex variable.

Proposition 2.5. *Let* $u : \Omega \longrightarrow [-\infty + \infty[$ *be a plurisubharmonic function. Fix* $a \in \Omega$ *and set* $\delta(a) := dist(a, \partial\Omega)$. *Then*

(*i*) *the spherical submean value inequality holds: for* $0 < r < \delta(a)$,

$$u(a) \leq \int_{|\xi|=1} u(a + r\xi)\, d\sigma(\xi), \tag{1.2}$$

where $d\sigma$ *is the normalized area measure on the unit sphere* $S^{2n-1} \subset \mathbb{C}^n$;

(*ii*) *the spatial submean value inequality holds: for* $0 < r < \delta(a)$ *and any increasing right continuous function* γ *on* $[0, r]$ *with* $\gamma(0) = 0$,

$$u(a) \leq \frac{1}{\gamma(r)} \int_0^r d\gamma(s) \int_{|\xi|=1} u(a + s\xi)\, d\sigma(\xi); \tag{1.3}$$

(*iii*) *the toric submean value inequality holds: for* $0 < r < \delta(a)/\sqrt{n}$,

$$u(a) \leq \int_{\mathbb{T}^n} u(a + r\zeta)\, d\tau_n(\zeta), \tag{1.4}$$

where $d\tau_n$ *is the normalized Lebesgue measure on the torus* $\mathbb{T}^n$.

All these integrals make sense in $[-\infty, +\infty[$. We will soon see that they are usually finite (cf Proposition 2.8).

Proof. The first inequality follows from (1.1) by integration over the unit sphere in $\mathbb{C}^n$ and the second inequality follows from the first one by integration over $[0,r]$ against the measure $d\gamma$. The third inequality follows from (1.1) by integration on the torus. □

Remark 2.6. *Let u be a plurisubharmonic in Ω. Using polar coordinates we can write*

$$\int_{|\zeta|<1} u(a+r\zeta)d\lambda(\zeta) = \int_0^r t^{2n-1}dt \int_{|\xi|=1} u(a+t\xi)d\sigma(\xi).$$

It follows from (1.3) that

$$u(a) \leq \frac{1}{\kappa_{2n}} \int_{|\zeta|<1} u(a+r\zeta)\, dV(\zeta), \tag{1.5}$$

where κ_{2n} denotes the volume of the unit ball in $\mathbb{C}^n$.

The function u considered as a function on $2n$ real variables is thus subharmonic in Ω considered as a domain in $\mathbb{R}^{2n}$.

Definition 2.7. *We denote by $PSH(\Omega)$ the convex cone of plurisubharmonic functions u in Ω such that $u|\Omega \not\equiv -\infty$.*

The submean-value inequalities imply the following important integrability result:

Proposition 2.8.

$$PSH(\Omega) \subset L^1_{loc}(\Omega).$$

Moreover the restriction of $u \in PSH(\Omega)$ to any euclidean sphere (resp. any torus $\mathbb{T}^n$) contained in Ω is integrable with respect to the area measure of the sphere (resp. the torus).

In particular the polar set $P(u) := \{u = -\infty\}$ has volume zero in Ω and its intersection with any euclidean sphere (resp. any torus $\mathbb{T}^n$) has measure zero with respect to the corresponding area measure.

Definition 2.9. *A set is called (locally) pluripolar if it is (locally) included in the polar set $\{u = -\infty\}$ of a function $u \in PSH(\Omega)$.*

It follows from previous proposition that pluripolar sets are somehow small. We will provide more precise information on their size in the next chapters.

Proof. Fix $u \in PSH(\Omega)$ et let G denote the set of points $a \in \Omega$ such that u is integrable in a neighborhood of a. We are going to show that G is a non empty open and closed subset of Ω. It will follow that $G = \Omega$ (by connectedness) and $u \in L^1_{loc}(\Omega)$.

Note that G is open by definition. If $a \in \Omega$ and $u(a) > -\infty$, the volume submean-value inequalities yield, for all $0 < r < \mathrm{dist}(a, \partial\Omega)$,

$$-\infty < \kappa_{2n} r^{2n} u(a) \leq \int_{\mathbb{B}(a,r)} u(z)\, dV(z).$$

Since u is bounded from above on $\mathbb{B}(a,r) \Subset \Omega$, it follows that u is integrable on $\mathbb{B}(a,r)$. In particular if $u(a) > -\infty$ then $a \in G$, hence $G \neq \emptyset$, since $u \not\equiv -\infty$.

We finally prove that G is closed. Let $b \in \Omega$ be a point in the closure of G and $r > 0$ so that $\mathbb{B}(b,r) \Subset \Omega$. By definition there exists $a \in G \cap \mathbb{B}(b,r)$. Since u is locally integrable in a neighborhood of a there exists a point a' close to a in $\mathbb{B}(b,r)$ such that $u(a') > -\infty$. Since $b \in \mathbb{B}(a',r) \Subset \Omega$ and u is integrable on $\mathbb{B}(a',r)$, it follows that $b \in G$.

The other properties are proved similarly, replacing volume submean inequalities by spherical (resp. toric) ones (Proposition 2.5). □

PROPOSITION 2.10. *Fix $u \in PSH(\Omega)$, $a \in \Omega$ and set $\delta_\Omega(a) :=$ dist(a, $\partial\Omega$). Fix γ a non decreasing right continuous function such that $\gamma(0) = 0$. Then*

$$r \longmapsto M_\gamma(a,r) := \frac{1}{\gamma(r)} \int_0^r d\gamma(s) \int_{|\xi|=1} u(a+s\xi)\, d\sigma(\xi),$$

is a non-decreasing continuous function in $[0, \delta_\Omega(a)[$ which converges to $u(a)$ as $r \to 0$.

PROOF. This property has already been established when $n = 1$. Fix $0 < r < \delta_\Omega(a)$ and observe that for all $e^{i\theta} \in \mathbb{T}$,

$$\int_{|\xi|=1} u(a+s\xi) d\sigma(\xi) = \int_{|\xi|=1} u(a+se^{i\theta}\cdot\xi) d\sigma(\xi),$$

since the area measure on the sphere is invariant under the action of the circle $\mathbb{T}$. Integrating on the circle and using the one-dimensional case yields the required property. □

COROLLARY 2.11. *For $u \in PSH(\Omega)$, $a \in \Omega$ and $0 < r < \delta_\Omega(a)$,*

$$u(a) \le \frac{1}{\kappa_{2n}} \int_{|\zeta|\le 1} u(a+r\zeta) d\lambda(\zeta) \le \int_{|\xi|=1} u(a+r\xi)\, d\sigma(\xi).$$

COROLLARY 2.12. *If two plurisubharmonic functions coincide almost everywhere, then they are equal.*

PROOF. Assume $u, v \in PSH(\Omega)$ are equal a.e. Then for all $a \in \Omega$,

$$\begin{aligned} u(a) &= \lim_{r\to 0} \frac{1}{\kappa_{2n}} \int_{|\zeta|\le 1} u(a+r\zeta) d\lambda(\zeta) \\ &= \lim_{r\to 0} \frac{1}{\kappa_{2n}} \int_{|\zeta|\le 1} v(a+r\zeta) d\lambda(\zeta) = v(a). \end{aligned}$$

□

We endow the space $PSH(\Omega)$ with the L^1_{loc}-topology. The following property will be used on several occasions in the sequel:

PROPOSITION 2.13. *The evaluation functional*

$$(u, z) \in PSH(\Omega) \times \Omega \longmapsto u(z) \in \mathbb{R} \cup \{-\infty\}$$

is upper semi-continuous.

In particular if $\mathcal{U} \subset PSH(\Omega)$ is a compact family of plurisubharmonic functions, its upper envelope

$$U := \sup\{u; u \in \mathcal{U}\}$$

is upper semi-continuous hence plurisubharmonic in Ω.

PROOF. Fix $(u, z_0) \in PSH(\Omega) \times \Omega$. Let (u_j) be a sequence in $PSH(\Omega)$ converging to u and let $r > 0$ and $\delta > 0$ be small enough.

We observe first that (u_j) is locally uniformly bounded from above. Indeed, if $B(a, 2r) \subset \Omega$, the submean value inequalities yield,

$$u_j(z) \leq \frac{1}{\kappa_{2n} r^{2n}} \int_{B(z,r)} u_j(w) dV(w) \leq \frac{1}{\kappa_{2n} r^{2n}} \int_{B(a,2r)} |u_j(w)| dV(w)$$

for $|z - a| < r$. Thus

$$\sup_{B(z_0,r)} u_j \leq \frac{1}{\kappa_{2n} r^{2n}} \int_{B(a,2r)} |u_j(w)| dV(w) \leq C,$$

since $\mathcal{U}$ is compact hence bounded.

We can thus assume without loss of generality that $u_j \leq 0$. The submean-value inequalities again yield, for $|z - z_0| < \delta$,

$$u_j(z) \leq \frac{1}{\kappa_{2n}(r + \delta)^{2n}} \int_{B(z,r+\delta)} u_j d\lambda \leq \frac{1}{\kappa_{2n}(r + \delta)^{2n}} \int_{B(z_0,r)} u_j d\lambda$$

Taking the limit in both j and z, we obtain

$$\limsup_{(j,z)\to(+\infty,z_0)} u_j(z) \leq \frac{1}{\kappa_{2n}(r + \delta)^{2n}} \int_{B(z_0,r)} u d\lambda.$$

Let $\delta \to 0^+$ and then $r \to 0^+$ to obtain

$$\limsup_{(j,z)\to(+\infty,z_0)} u_j(z) \leq u(z_0),$$

which proves the desired semi-continuity at point (u, z_0).

It follows that the envelope U is upper semi-continuous. Since it clearly satisfies the mean value inequalities on each complex line, we infer that U is plurisubharmonic . □

The upper envelope of a family of plurisubharmonic functions which is merely relatively compact is not necessarily upper semi-continuous. It turns out that its upper semi-continuous regularization is plurisubharmonic .

To study such envelopes, we need the following classical topological lemma of Choquet which we will use on several occasions.

LEMMA 2.14. *Let $\mathcal{U}$ be a family of upper semi-continuous functions and let $U := \sup\{u; u \in \mathcal{U}\}$ be its upper envelope. There exists a countable sub-family (u_j) in $\mathcal{U}$ such that $U^* = (\sup_j u_j)^*$ in Ω, where*

$$U^*(z) = \limsup_{z' \to z} U(z').$$

PROOF. Assume first that $\mathcal{U}$ is locally uniformly bounded from above. Let $B(z_j, r_j)$ be a countable basis for the topology of Ω. For each j let $z_{j,k}$ a sequence in the ball $B(z_j, r_j)$ such that

$$\sup_{B(z_j,r_j)} U = \sup_k U(z_{j,k}).$$

For each (j,k) there exists a sequence $(u^\ell_{j,k})_{\ell \in \mathbb{N}}$ in $\mathcal{U}$ such that

$$U(z_{j,k}) = \sup_\ell u^\ell_{j,k}(z_{j,k}).$$

Set $V := \sup_{j,k,\ell} u^\ell_{j,k}$. Then $V \le U$ hence $V^* \le U^*$. Now

$$\sup_{B(z_j,r_j)} V \ge \sup_k V(z_{j,k}) \ge \sup_{k,\ell} u^\ell_{j,k}(z_{j,k}) = \sup_k U(z_{j,k}) = \sup_{B(z_j,r_j)} U,$$

hence $\sup_{B(z_j,r_j)} V = \sup_{B(z_j,r_j)} U$ for all j. Since any $B(z,\varepsilon)$ is a union of balls $B(z_j, r_j)$, we get $\sup_{B(z,\varepsilon)} V = \sup_{B(z,\varepsilon)} U$ hence $V^*(z) = U^*(z)$.

To treat the general case we define, for $u \in \mathcal{U}$, $\tilde{u} := \chi \circ u$, where $\chi : t \in \mathbb{R} \mapsto t/(1+|t|) \in]-1, +1[$ is an increasing homeomorphism. Observe that

$$\tilde{U} = \chi \circ U \text{ and } \tilde{U}^* = \chi \circ U^*$$

hence the conclusion follows from the previous case applied to $\tilde{U}$. □

The following result is du to P. Lelong ([**Lel57, Lel67**]).

PROPOSITION 2.15. *Let $(u_i)_{i \in I}$ be a family of plurisubharmonic functions in a domain Ω, which is locally uniformly bounded from above in Ω and let $u := \sup_{i \in I} u_i$ be its upper envelope. The usc regularization*

$$z \mapsto u^*(z) := \limsup_{\Omega \ni z' \to z} u(z') \in \mathbb{R} \cup \{-\infty\}$$

is plurisubharmonic in Ω and $\{u < u^*\}$ has Lebesgue measure zero.

PROOF. It follows from Choquet Lemma 2.16 that there exists an increasing sequence $v_j = u_{i_j}$ of plurisubharmonic functions such that

$$u^* = (\lim v_j)^*.$$

Set $v = \lim \nearrow v_j$. This function satisfies various mean-value inequalities but it is not necessarily upper semi-continuous. Let χ_ε be standard radial mollifiers. Observe that, for $\varepsilon > 0$ fixed, $v_j * \chi_\varepsilon$ is an increasing sequence of plurisubharmonic functions, thus its continuous limit $v * \chi_\varepsilon$ is plurisubharmonic and $\varepsilon \mapsto v * \chi_\varepsilon$ is increasing.

We let w denote the limit of $v * \chi_\varepsilon$ as ε decreases to zero. The function w is plurisubharmonic as a decreasing limit of plurisubharmonic functions. It satisfies, for all $\varepsilon > 0$, $w \le u * \chi_\varepsilon$ since $v_j * \chi_\varepsilon \le u * \chi_\varepsilon$.

On the other hand for all $\varepsilon > 0$, $u \leq v^* \leq v * \chi_\varepsilon$ hence

$$u \leq u^* = v^* \leq w \leq u * \chi_\varepsilon.$$

Since $u * \chi_\varepsilon$ converges to u in L^1_{loc}, we conclude that $u^* = w$ is plurisubharmonic. Note that the set $\{u < u^*\}$ has Lebesgue measure zero. $\square$

We have used the following elementary topological lemma due to Choquet:

LEMMA 2.16. *Let $\mathcal{U}$ be a family of upper semi-continuous functions and let $U := \sup\{u; u \in \mathcal{U}\}$ be its upper envelope. Then there exists a countable sub-family (u_j) in $\mathcal{U}$ such that $U^* = (\sup_j u_j)^*$ in Ω, where*

$$U^*(z) = \limsup_{z' \to z} U(z').$$

PROOF. Assume first that $\mathcal{U}$ is locally uniformly bounded from above. Let $B(z_j, r_j)$ be a countable basis for the topology of Ω. For each j let $z_{j,k}$ a sequence in the ball $B(z_j, r_j)$ such that

$$\sup_{B(z_j,r_j)} U = \sup_k U(z_{j,k}).$$

For each (j,k) there exists a sequence $(u^\ell_{j,k})_{\ell \in \mathbb{N}}$ in $\mathcal{U}$ such that

$$U(z_{j,k}) = \sup_\ell u^\ell_{j,k}(z_{j,k}).$$

Set $V := \sup_{j,k,\ell} u^\ell_{j,k}$. Then $V \leq U$ hence $V^* \leq U^*$. Now

$$\sup_{B(z_j,r_j)} V \geq \sup_k V(z_{j,k}) \geq \sup_{k,\ell} u^\ell_{j,k}(z_{j,k}) = \sup_k U(z_{j,k}) = \sup_{B(z_j,r_j)} U,$$

hence $\sup_{B(z_j,r_j)} V = \sup_{B(z_j,r_j)} U$ for all j. Since any $B(z,\varepsilon)$ is a union of balls $B(z_j, r_j)$, we get $\sup_{B(z,\varepsilon)} V = \sup_{B(z,\varepsilon)} U$ hence $V^*(z) = U^*(z)$.

To treat the general case we define, for $u \in \mathcal{U}$, $\tilde{u} := \chi \circ u$, where $\chi : t \in \mathbb{R} \mapsto t/(1+|t|) \in]-1,+1[$ is an increasing homeomorphism. Observe that

$$\tilde{U} = \chi \circ U \text{ and } \tilde{U}^* = \chi \circ U^*$$

hence the conclusion follows from the previous case applied to $\tilde{U}$. $\square$

EXAMPLE 2.17. *If $f : \Omega \longrightarrow \mathbb{C}$ is a holomorphic function such that $f \not\equiv 0$ and $c > 0$ then $c \log|f| \in PSH(\Omega)$. Conversely one can show, when Ω is pseudoconvex, that the cone $PSH(\Omega)$ is the closure (in $L^1_{loc}(\Omega)$) of the set of functions*

$$\{c \log|f|; f \in \mathcal{O}(\Omega), f \not\equiv 0, c > 0\}.$$

This can be shown by using Hörmander's L^2-estimates [**Hor90**].

1.3. Differential characterization. We show in this section that plurisubharmonicity can be characterized by (weak) differerential inequalities.

1.3.1. *Plurisubharmonic smoothing.* We first explain how any $u \in PSH(\Omega)$ can be approximated by a decreasing family of smooth plurisubharmonic functions (on any subdomain $D \Subset \Omega$).

Let $\rho(z) \geq 0$ be a smooth radial function on $\mathbb{C}^n$ with compact support in the unit ball $\mathbb{B} \subset \mathbb{C}^n$ such that $\int_{\mathbb{C}^n} \rho(z) d\lambda(z) = 1$. We set

$$\rho_\varepsilon(\zeta) := \varepsilon^{-2n} \rho(\zeta/\varepsilon),$$

for $\varepsilon > 0$. The functions ρ_ε are smooth with compact support in $\mathbb{B}(0, \varepsilon)$ and $\int_{\mathbb{C}^n} \rho_\varepsilon d\lambda = 1$, they approximate the Dirac mass at the origin.

Let $u : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a L^1_{loc}-function. We set

$$\Omega_\varepsilon := \{z \in \Omega; \mathrm{dist}(z, \partial\Omega) > \varepsilon\}$$

and consider, for $z \in \Omega_\varepsilon$,

$$u_\varepsilon(z) := \int_\Omega u(\zeta) \rho_\varepsilon(z - \zeta) d\lambda(\zeta).$$

These functions are smooth and converge to u in L^1_{loc}.

Proposition 2.18. *If $u \in PSH(\Omega)$ then the smooth functions u_ε are plurisubharmonic and decrease to u as ε decreases to 0^+.*

Proof. The functions u_ε are plurisubharmonic as (convex) average of plurisubharmonic functions. By definition if $z \in \Omega_\varepsilon$, we have

$$u_\varepsilon(z) = \int_{|\zeta|<1} u(z + \varepsilon\zeta) \rho(\zeta) d\lambda(\zeta), \ z \in \Omega_\varepsilon.$$

Integrating in polar coordinates we get

$$u_\varepsilon(z) = \int_0^1 r^{2n-1} \rho(r) dr \int_{|\xi|=1} u(z + \varepsilon r \xi) d\sigma(\xi).$$

The monotonicity property now follows from Proposition 2.10. $\square$

We let the reader check that a function u of class C^2 is plurisubharmonic in Ω iff for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$,

$$\mathcal{L}_u(a; \xi) := \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}(a) \xi_i \overline{\xi}_j \geq 0. \tag{1.6}$$

In other words the hermitian form $\mathcal{L}_u(a, .)$ (the *Levi form* of u at the point a) should be semi-positive in $\mathbb{C}^n$.

For non smooth plurisubharmonic functions this positivity condition has to be understood in the sense of distributions:

Proposition 2.19. *If $u \in PSH(\Omega)$ then for any $\xi \in \mathbb{C}^n$,*

$$\sum_{1 \leq j,k \leq n} \xi_j \bar{\xi}_k \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \geq 0$$

is a positive distribution in Ω.

Conversely if $U \in \mathcal{D}'(\Omega)$ is a distribution such that for all $\xi \in \mathbb{C}^n$, the distribution $\sum \xi_j \bar{\xi}_k \frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}$ is positive, then there exists a unique $u \in PSH(\Omega)$ such that $U \equiv T_u$.

Each distribution $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ extends to a complex Borel measure $\mu_{j,\bar{k}}$ on Ω so that the matrix $(\mu_{j,\bar{k}})$ is hermitian semi-positive.

Proof. The proof follows that of Proposition 1.25. Fix $\xi \in \mathbb{C}^n$ and consider the linear operator with constant coefficients

$$\Delta_\xi := \sum_{1 \le j,k \le n} \xi_j \bar{\xi}_k \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.$$

Assume first that u is smooth in Ω and fix $a \in \Omega$. The one variable function $u_\xi : \zeta \longmapsto u(a + \zeta \cdot \xi)$ is defined on a small disc around 0. For $\zeta \in \mathbb{C}$ small enough observe that

$$\frac{\partial^2 u_\xi}{\partial \zeta \partial \bar{\zeta}}(\zeta) = \Delta_\xi u(a + \zeta \cdot \xi).$$

This yields the first claim of the proposition in this smooth setting.

We proceed by regularization to treat the general case. Since Δ_ξ is linear with constant coefficients, it commutes with convolution,

$$\Delta_\xi u_\varepsilon = (\Delta_\xi u)_\varepsilon$$

and we can pass to the limit to conclude.

For the converse we proceed as in the proof Proposition 1.25 to show that the distributions $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}$ are non-negative in Ω. Thus they extend into non-negative Borel measures in Ω. The mixed complex derivatives are controlled by using polarization identities for hermitian forms. □

1.3.2. *Invariance properties.* Plurisubharmonicity is invariant under holomorphic changes of coordinates, hence it makes sense on complex manifolds. More generally we have the following:

Proposition 2.20. *Fix $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$. If $u \in PSH(\Omega)$ and $f : \Omega' \longrightarrow \Omega$ is a holomorphic map, then $u \circ f \in PSH(\Omega')$.*

Proof. Using convolutions we reduce to the case when u is smooth. Fix $z_0 \in \Omega'$ and set $w_0 = f(z_0)$, by the chain rule we get

$$\frac{\partial^2 u \circ f}{\partial z_i \partial \bar{z}_j}(z_0) = \sum_{1 \le k,l \le n} \frac{\partial f_k}{\partial z_i} \overline{\frac{\partial f_l}{\partial z_j}}(z_0) \frac{\partial^2 u}{\partial w_k \partial \bar{w}_l}(w_0),$$

for $i, j = 1, \cdots, m$. Thus for all $\xi \in \mathbb{C}^n$,

$$\sum_{i,j} \xi_j \bar{\xi}_j \frac{\partial^2 u \circ f}{\partial z_i \partial \bar{z}_j}(z_0) = \sum_{k,l} \eta_k \bar{\eta}_l \frac{\partial^2 u}{\partial w_k \partial \bar{w}_l}(w_0),$$

where $\eta_k := \sum_{i=1}^m \xi_i \frac{\partial f_k}{\partial z_i}(z_0)$ for $k = 1, \cdots, n$.

This means in terms of the Levi forms that $\mathcal{L}_{u\circ f}(z_0;\xi) = \mathcal{L}_u(w_0;\eta)$, where $\eta := \partial f(z_0)\cdot\xi$ and $w_0 := f(z_0)$. The result follows. □

We have observed that plurisubharmonic functions are subharmonic functions when considered as functions of $2n$ real variables, identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Conversely one can characterize plurisubharmonic functions as those subharmonic functions in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ which are invariant under complex linear transformations of $\mathbb{C}^n$:

PROPOSITION 2.21. *Let $\Omega \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ be a domain and $u : \Omega \to [-\infty, +\infty[$ an upper semi-continuous function in Ω. Then u is plurisubharmonic in Ω iff for any complex affine transformation $S : \mathbb{C}^n \to \mathbb{C}^n$, $u \circ S$ is subharmonic in the domain $S^{-1}(\Omega) \subset \mathbb{R}^{2n}$.*

PROOF. One implication follows from Proposition 2.20: if u is plurisubharmonic, then $u \circ S$ is subharmonic for any complex affine trasnformation $S : \mathbb{C}^n \to \mathbb{C}^n$.

We now prove the converse. Let $r > 0$ be small enough and $z \in \Omega_r$. By assumption for all $0 < \varepsilon < 1$, the function $\xi \mapsto u(z_1 + r\xi_1, z' + r\varepsilon\xi')$ is subharmonic in ξ in a neighborhood of the unit sphere $\{|\xi| = 1\}$ in $\mathbb{R}^{2n}$. The spherical submean value inequality yields

$$u(z) \leq \int_{|\xi|=1} u(z_1 + r\xi_1, z' + r\varepsilon\xi')d\sigma(\xi),$$

where $d\sigma$ denotes the normalized Lebesgue measure on the unit sphere. Since u is upper semi-continuous and locally bounded from above, by Fatou's lemma, we obtain as $\varepsilon \to 0$

$$u(z) \leq \int_{|\xi|=1} u(z_1 + r\xi_1, z')d\sigma(\xi).$$

This means that the function of one complex variable $\zeta \longrightarrow u(\zeta, z')$ is subharmonic in its domain. The subharmonicity on other lines follows from the invariance under complex transformations. □

2. Hartogs lemma and the Montel property

2.1. Hartogs lemma.

THEOREM 2.22. *Let (u_j) be a sequence of functions in $PSH(\Omega)$ which is locally uniformly bounded from above in Ω.*

1. If (u_j) does not converge to $-\infty$ locally uniformly on Ω, then it admits a subsequence which converges to some $u \in PSH(\Omega)$ in $L^1_{loc}(\Omega)$.

2. If $u_j \to U$ in $\mathcal{D}'(\Omega)$ then the distribution U is defined by a unique function $u \in PSH(\Omega)$. Moreover

- *$u_j \to u$ in $L^1_{loc}(\Omega)$*
- *$\limsup u_j(z) \leq u(z)$ for all $z \in \Omega$, with equality a.e. in Ω.*

- *for any compact set K and any continuous function h on K,*

$$\limsup \max_K (u_j - h) \leq \max_K (u - h).$$

The last item is usually called *Hartogs lemma*. When the compact set K is "regular", we actually have an equality,

$$\lim \max_K (u_j - h) = \max_K (u - h).$$

We are not going to study this notion any further here. The reader will check in Exercise 2.9 that if a compact set is the closure of an open set with smooth boundary, then it is regular.

Proof. The statement is local so we can assume that $\Omega \Subset \mathbb{C}^n$ and $u_j \leq 0$ in Ω for all $j \in \mathbb{N}$ (substracting a constant if necessary).

Since (u_j) does not converge uniformly towards $-\infty$, we can find a compact set E and $C > 0$ such that

$$\limsup_{j \to +\infty} \max_E u_j \geq -C > -\infty.$$

Thus there exists an increasing sequence (j_k) of integers and a sequence of points (x_k) in E such that the sequence $u_{j_k}(x_k)$ is bounded from below by $-2C$. Extracting again we can assume that $x_k \to a \in E$. Set for simplicity $v_k := u_{j_k}$ for $k \in \mathbb{N}$.

We know that $v_k \in L^1_{loc}(\Omega)$ and we claim that if $B \Subset \Omega$ is a ball around a, the sequence $\int_B v_k d\lambda$ is bounded. Indeed for k large enough there is a ball B_k centered at x_k such that $B \subset B_k \Subset \Omega$ hence

$$\int_B v_k d\lambda \geq \int_{B_k} v_k d\lambda \geq \lambda(B_k) v_k(x_k) \geq -2C\lambda(\Omega).$$

By the same reasoning as in the proof of Proposition 2.8, we deduce from this that the set X of points $x \in \Omega$ which have a neighborhood $W \subset \Omega$ such that the sequence $\int_W v_k d\lambda$ is bounded below is closed. Since it is open (by definition) and not empty (by assumption), we infer from connectedness that $X = \Omega$.

The sequence (v_k) is therefore bounded in $L^1_{loc}(\Omega)$, i.e. the sequence of non-negative measures $\mu_k := (-v_k)\lambda$ is bounded in the weak topology of Radon measures on Ω. Thus it admits a subsequence which converges weakly (in the sense of Radon measures), hence the first assertion is now a consequence of the second one.

We now prove the second statement. Assume that $u_j \rightharpoonup U$ in the weak sense of distributions on Ω. It follows from Proposition 2.19 that $U = T_u$ is defined by a plurisubharmonic function u.

We want to show that $u_j \to u$ in $L^1_{loc}(\Omega)$. Fix (ρ_ε) mollifiers as earlier. We observe that the sequence $(u_j \star \rho_\varepsilon)_{j \in \mathbb{N}}$ is equicontinuous in Ω_ε since (u_j) is bounded in $L^1_{loc}(\Omega)$: indeed fix $a \in \Omega_\varepsilon$, $0 < \eta < \varepsilon/2$, then for $x, y \in B(a, \eta)$,

$$|u_j \star \rho_\varepsilon(x) - u_j \star \rho_\varepsilon(y)| \leq \sup_{|t| \leq \varepsilon} |\rho(x-t) - \rho(y-t)| \cdot \|u_j\|_{L^1(B(a,\varepsilon))}.$$

Fix $K \subset \Omega$ a compact set and χ a continuous test function in Ω such that $\chi \equiv 1$ on K and $0 \leq \chi \leq 1$ on Ω. Then

$$\begin{aligned} \int_K |u_j - u| d\lambda \ &\leq \ \int (u_j \star \rho_\varepsilon - u_j)\chi d\lambda + \int \chi |u_j \star \rho_\varepsilon - u \star \rho_\varepsilon| d\lambda \\ &+ \ \int (u \star \rho_\varepsilon - u)\chi d\lambda. \end{aligned}$$

We use here the key fact that $u_j \star \rho_\varepsilon - u_j \geq 0$.

By weak convergence the fist term converges to $\int (u \star \rho_\varepsilon - u)\chi d\lambda$ and by equicontinuity, $u_j \star \rho_\varepsilon \longrightarrow u \star \rho_\varepsilon$ uniformly on K as $j \to +\infty$. Hence

$$\limsup_{j \to +\infty} \int_K |u_j - u| d\lambda \leq 2 \int (u \star \rho_\varepsilon - u)\chi d\lambda.$$

The monotone convergence theorem insures that the right hand side converges to 0 as $\varepsilon \searrow 0$.

Since $u_j \star \rho_\varepsilon \longrightarrow u \star \rho_\varepsilon$ locally uniformly in Ω_ε and $u_j \leq u_j \star \rho_\varepsilon$, it follows that $\limsup u_j \leq u \star \rho_\varepsilon$ in Ω, hence $\limsup u_j \leq u$ in Ω.

Fatou's lemma insures that for any fixed compact set $K \subset \Omega$,

$$\int_K u d\lambda = \lim_j \int_K u_j d\lambda \leq \int_K (\limsup_j u_j) d\lambda \leq \int_K u d\lambda$$

As $u - \limsup u_j \geq 0$ in Ω and $\int_K (u - \limsup u_j) d\lambda = 0$, we infer $u - \limsup_j u_j = 0$ almost everywhere in K.

To prove the last property, we observe that

$$\max_K (u_j - h) \leq \max_K (u_j \star \rho_\varepsilon - h) \to \max_K (u \star \rho_\varepsilon - h),$$

where the last convergence follows from the equicontinuity of the family $(u_j \star \rho_\varepsilon - h)$ for fixed $\varepsilon > 0$. □

The following consequence is a kind of "Montel property" of the convex set $PSH(\Omega)$:

Corollary 2.23. *The space $PSH(\Omega)$ is a closed subset of $L^1_{loc}(\Omega)$ for the L^1_{loc}-topology which has the Montel property: every bounded subset in $PSH(\Omega)$ is relatively compact.*

2.2. Comparing topologies. Plurisubharmonic functions have rather good integrability properties: they belong to the spaces L^p_{loc} for all $1 \leq p < +\infty$ and their gradient are in L^q_{loc} for all $1 \leq q < 2$:

Theorem 2.24. *Let (u_j) be a sequence of functions in $PSH(\Omega)$ converging in L^1_{loc} to $u \in PSH(\Omega)$. Then*

1. *the sequence is locally uniformly bounded from above;*
2. *$u_j \to u$ in L^p_{loc} for all $p \geq 1$;*

(3) *the gradients* Du_j *converge in* L^q_{loc} *to* Du *for all* $q < 2$.

Together with Theorem 2.22, this result shows that

$$PSH(\Omega) \subset L^p_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$$

and the weak topology of distributions on Ω and the L^p_{loc}-topology coincide on the space $PSH(\Omega)$ for all $p \geq 1$.

PROOF. *Step 1.* We first show that (u_j) is locally uniformly bounded in L^p_{loc}, for all $p \geq 1$. Assume first that $n = 1$, $\bar{\mathbb{D}} \subset \Omega$ and $u(0) > -\infty$. We can assume without loss of generality that $u \leq 0$. The Poisson-Jensen formula yields

$$u(z) = \int_{\partial\mathbb{D}} u(\zeta) \frac{1 - |z|^2}{|z - \zeta|^2} d\sigma(\zeta) + \int_{|\zeta|<1} \log \frac{|z - \zeta|}{|1 - z\bar{\zeta}|} d\mu(\zeta), \tag{2.1}$$

where $d\sigma(\zeta) := \frac{|d\zeta|}{2\pi}$ is the normalized length measure on $\partial\mathbb{D}$ and $\mu := \frac{\Delta u}{2\pi}$ is the Riesz measure of u in $\mathbb{D}$. In particular

$$u(0) = \int_0^{2\pi} u(e^{i\theta}) \frac{d\theta}{2\pi} + \int_{|\zeta|<1} \log |\zeta| d\mu(\zeta).$$

Set

$$h(z) = \int_{\partial\mathbb{D}} u(\zeta) \frac{1 - |z|^2}{|z - \zeta|^2} d\sigma(\zeta).$$

This is a negative harmonic function in the unit disc. It follows from Harnack inequalities that $3h(0) \leq h(z) \leq 3^{-1}h(0)$ for $|z| < 1/2$. Thus for $p \geq 1$,

$$\left(\int_{|z|<1/2} |h(z)|^p d\lambda(z) \right)^{1/p} \leq 3(\pi/4)^{1/p} |h(0)|.$$

We claim that there is $C_p > 0$ such that for all $a \in \mathbb{D}$,

$$\left(\int_{|z|<1/2} \left(-\log \frac{|z - a|}{|1 - z\bar{a}|} \right)^p d\lambda(z) \right)^{1/p} \leq -C_p \log |a|. \tag{2.2}$$

Indeed set $h_a(z) := -\log \frac{|z-a|}{|1-z\bar{a}|}$. This is a positive harmonic function in $\mathbb{D} \setminus \{a\}$ with a logarithmic singularity at a. Moreover

$$0 \leq h_a(z) \leq \log \frac{2}{|z - a|},$$

for $|z| < 1$, hence for $|a| < 1$,

$$\int_{|z|<1/2} |h_a(z)|^p d\lambda(z) \leq \Gamma(p + 1).$$

The inequality (2.2) is thus valid when $|a| \leq 3/4$ with C_p such that

$$C_p \geq \Gamma(p + 1)^{1/p} / (\log(4/3))^p.$$

Assume now $|a| > 3/4$. Then h_a is a positive harmonic function near $\overline{D}(3/4)$. It follows again from Harnack inequalities that

$$0 \leq h_a(z) \leq 5h_a(0) = 5\log(1/|a|),$$

for $|z| \leq 1/2$. Therefore

$$\int_{|z|<1/2} |h_a(z)|^p d\lambda(z) \leq 5^p \left(\log(1/|a|)\right)^p.$$

This proves our claim with $C_p := \max\{\Gamma(p+1)^{1/p}/(\log(4/3)), 5\}$. It follows now from Minkowski's inequality that

$$\left(\int_{|z|<1/2} |u|^p d\lambda(z)\right)^{1/p} \leq C_p(|h(0)| + \int_{|\zeta|<1} \log(1/|\zeta|) d\mu(\zeta) = C_p|u(0)|.$$

In higher dimension we use this inequality n-times: if u is plurisubharmonic, $u \leq 0$ near $a + \bar{\mathbb{D}}^n_R \subset \Omega$, and $u(a) > -\infty$ then

$$\int_{\mathbb{D}^n_{1/2}} |u(a+Rz)|^p d\lambda(z) \leq C_p^{np} |u(a)|^p. \tag{2.3}$$

Using the same reasonning as in the proof of Proposition 2.8 we deduce from (2.3) that the set of points where $|u|^p$ is locally integrable is a non empty open and closed set in Ω, hence $u \in L^p_{loc}(\Omega)$.

Recall now that $u_j \to u$ in L^1_{loc}, thus (u_j) does not converge uniformly to $-\infty$ on any compact set $K \Subset \Omega$ and it is locally bounded from above in Ω. Arguing as in the proof of Proposition 2.8 we infer from (2.3) that the set of point where the sequence $|u_j|^p$ is locally uniformly integrable is a non empty open and closed set in Ω, hence (u_j) is locally bounded in $L^p_{loc}(\Omega)$.

Step 2. We now show that $u_j \to u$ in $L^p_{loc}(\Omega)$. Fix a compact set $K \Subset \Omega$ and assume that $u_j \leq 0$ in K for all $j \in \mathbb{N}$.

Assume first that the sequence is locally uniformly bounded. There exists $M > 0$ such that for any $j \in \mathbb{N}$, $-M \leq u_j \leq 0$ on K. We fix a subsequence (v_j) of (u_j) such that $v_j \to u$ almost everywhere. Lebesgue convergence theorem insures that $v_j \to u$ in $L^p(K)$. This implies that u is the unique limit point of the sequence (u_j) in $L^p_{loc}(\Omega)$.

To treat the general case we set for $m \geq 1$ and $j \in \mathbb{N}$

$$u_j^m := \sup\{u_j, -m\}, \quad u^m := \sup\{u, -m\}.$$

Minkowski's inequality yields

$$\|u_j - u\|_{L^p(K)} \leq \|u_j - u_j^m\|_{L^p(K)} + \|u_j^m - u^m\|_{L^p(K)} + \|u^m - u\|_{L^p(K)}.$$

By the monotone convergence theorem, the last term converges to 0 as $m \to +\infty$. By the previous case, for a fixed m the second term converges to 0 as $j \to +\infty$. To conclude it is thus enough to show that the first term converges to 0 uniformly in j as $m \to +\infty$. Markov inequality yields, for $m \geq 1$ and $j \in \mathbb{N}$,

$$\begin{aligned}\int_K |u_j - u_j^m|^p d\lambda &= 2\int_{K\cap\{u_j\leq -m\}} |u_j|^p d\lambda \\ &\leq \frac{2}{m}\int_K |u_j|^{p+1} d\lambda,\end{aligned}$$

which allows to conclude since (u_j) is bounded in $L^{p+1}(K)$.

Step 3. We now establish local uniform bounds on the gradient of u in $L^p(\Omega)$ for $1 \leq p < 2$. Assume first that $n = 1$. It suffices to consider the case when $2\mathbb{D} \Subset \Omega$ and $u(0) > -\infty$ and get a uniform estimate on $\mathbb{D}_{1/2}$.

The Poisson-Jensen formula (2.1) shows that for $z \in \mathbb{D}$,

$$2\partial_z u(z) = 2\partial_z h(z) + \int_{\mathbb{D}} \frac{1-|\zeta|^2}{(z-\zeta)(1-z\bar{\zeta})} d\mu(\zeta).$$

Since h is harmonic, the representation formula yields

$$\partial_z h(z) = \int_{\partial\mathbb{D}} u(\zeta)\partial_z P(z,\zeta) d\sigma(\zeta)$$

when $z \in \mathbb{D}$. Since

$$\partial_z P(z,\zeta) = \frac{-\bar{z}}{|z-\zeta|^2} - \frac{(|1-|z|^2)(\bar{\zeta}-\bar{z})}{|z-\zeta|^4},$$

it follows that for $|z| \leq 1/2$,

$$|\partial_z h(z)| \leq 6\int_{\partial\mathbb{D}} |u(\zeta)| d\sigma(\zeta) \leq 6\|u\|_{L^1(\mathbb{D})}.$$

We have used here (2.11) and the fact that $u \leq 0$.

We need a uniform estimate for the second term which we denote by $g(z)$. From the expression of g we get

$$|g(z)| \leq 2\int_{\mathbb{D}} \frac{d\mu(\zeta)}{|z-\zeta|}.$$

Using Minkowski's inequality we deduce that

$$\begin{aligned}\|g\|_{L^p(\{|z|<1/2\})} &\leq 2\int_{\mathbb{D}} \left(\int_{\{|z|<1/2\}} \frac{d\lambda(z)}{|z-\zeta|^p}\right)^{1/p} d\mu(\zeta) \\ &\leq 2\pi\frac{2^{2-p}}{2-p}\int_{\mathbb{D}} d\mu.\end{aligned}$$

Since $2\mathbb{D} \Subset \Omega$ we apply Stokes' formula to get

$$\int_{\mathbb{D}} d\mu = \int_{\mathbb{D}} dd^c u \leq \frac{1}{3}\int_{\mathbb{D}} (4-|z|^2) dd^c u \leq \frac{1}{3}\int_{2\mathbb{D}} (-u) d\lambda(z).$$

Adding all these inequalities, we get a uniform bound on the gradient of u in the disc $|z| \leq 1/2$,

$$\|\partial_z u\|_{L^p(\frac{1}{2}\mathbb{D})} \leq c_p \|u\|_{L^1(2\mathbb{D})},$$

where c_p is a uniform constant depending only on p.

Using these inequalities n times we get a local uniform bound for the gradient of any function $u \in PSH(\Omega)$. In particular for any $p < 2$, we have $PSH(\Omega) \subset W^{1,p}_{loc}(\Omega)$ and the inclusion operator takes bounded sets onto bounded sets.

Step 4. We finally prove that this inclusion is continuous. Let $(u_j) \in PSH(\Omega)^{\mathbb{N}}$ be a sequence converging to u in L^1_{loc}. Since plurisubharmonic functions are subharmonic in $\mathbb{R}^{2n}$, it follows from Exercise 2.10 that $Du_j \to Du$ in L^1_{loc}, hence almost everywhere (up to extracting and relabelling). The local uniform bounds for $||Du_j||_{L^q}$, $q < 2$, allow to conclude as above that $Du_j \to Du$ in L^q_{loc} for all $q < 2$. □

3. Currents in the sense of de Rham

We let in this section Ω denote an open subset of $\mathbb{R}^N$ $N \geq 2$.

3.1. Forms with distributions coefficients. A differential form α of degree p on Ω with locally integrable coefficients

$$\alpha = \sum_{|I|=p} \alpha_I dx_I,$$

acts as a linear form on the space of continuous test forms of complementary degree $q = N - p$: if $\psi = \chi dx_K$ is a continuous test form of degree q with compact support then

$$< \alpha, \psi >= \sum_{|I|=p} \varepsilon_{I,K} \int_\Omega \chi \alpha_I \, dV,$$

where $\varepsilon_{I,K}$ is such that $dx_I \wedge dx_K = \varepsilon_{I,K} \, dV$, with $\varepsilon_{I,K} = 0$ unless $K = I^c$ complements I in $[1, N]$ in which case $\varepsilon_{I,K} = \pm 1$.

DEFINITION 2.25. *A current S of degree p is a continuous linear form on the space $\mathcal{D}_{N-p}(\Omega)$ of test forms (i.e. smooth differential forms with compact support) of degree $N - p$ on Ω.*

We let $\mathcal{D}'_{N-p}(\Omega)$ denote the space of currents of degree p. The action of S on a test form $\Psi \in \mathcal{D}_{N-p}(\Omega)$ is denoted by $< S, \Psi >$.

If α is a smooth fom of degree q the wedge product of α and S is defined as follows:

DEFINITION 2.26. *For a test form Ψ of degree $N - p - q$, we set*

$$< S \wedge \alpha, \Psi >:=< S, \alpha \wedge \Psi > .$$

We define similarly $\alpha \wedge S := (-1)^{pq} S \wedge \alpha$.

Observe that the current $S \wedge \alpha \wedge \Psi$ is a current of maximal degree with compact support, it can be identified with a distribution with compact support. One can similarly interpret a current of degree p as a differential form of degree p with distribution coefficients.

3.2. Closed currents. When S is a smooth form of degree p in Ω and Ψ is a test form of degree $N-p-1$,

$$d(S \wedge \Psi) = dS \wedge \Psi + (-1)^p S \wedge d\Psi.$$

Since $S \wedge \Psi$ is a differential form with compact support, it follows from Stokes formula that $\int_\Omega d(S \wedge \Psi) = 0$ hence

$$\int_\Omega dS \wedge \Psi = (-1)^{p+1} \int_\Omega S \wedge d\Psi$$

This suggests the following definition:

DEFINITION 2.27. *If S is a current of degree p then dS is the current of degree $(p+1)$ defined by*

$$< dS, \psi >= (-1)^{p+1} < S, d\psi >,$$

where ψ is any test form of degree $N-p-1$.

This definition allows one to extend differential calculus on forms to currents. It follows from the definition that a current of degree p is a differential p-form $T = \sum_{|I|=p} T_I dx_I$, with distribution coefficients. We set

$$dT = \sum_{|I|=p} \sum_{1 \le j \le N} \frac{\partial T_I}{\partial x_j} dx_j \wedge dx_I$$

where $\frac{\partial T_I}{\partial x_j}$ is the partial derivative of the distribution T_I acting on test functions according to Stokes formula by

$$< \frac{\partial T_I}{\partial x_j}, \psi >= - < T_I, \frac{\partial \psi}{\partial x_j} > .$$

The reader can check that this is consistent with the above definition of dT.

Most properties valid in the differential calculus on forms extend to currents. In particular if T is a current of degree p and α is a differential form of degree m, then

$$d(T \wedge \alpha) = dT \wedge \alpha + (-1)^p T \wedge d\alpha,$$

as the reader may check.

The following version of Stokes' formula for currents will be used on several occasions:

LEMMA 2.28. *Let S be a current of degree $N-1$ with compact support in Ω. Then*

$$\int_\Omega dS = 0.$$

PROOF. Let χ be a smooth cut off function in Ω such that $\chi \equiv 1$ in a neighborhood of K, a compact subset of Ω containing the support of S. Then

$$\int_\Omega dS = \int_\Omega \chi dS =< S, d\chi >= 0,$$

since $d\chi = 0$ in a neighborhood of the support of S. □

3.3. Bidegree. Assume now $\Omega \subset \mathbb{C}^n$ is a domain in the complex hermitian space $\mathbb{C}^n$. The complex structure induces a splitting of differential forms into types. The space of test forms of bidegree (q, q) will be denoted by $\mathcal{D}_{q,q}(\Omega)$, where $0 \leq q \leq n$.

DEFINITION 2.29. *A current T of bidegree (p, p) is a differential form of bidegree (p, p) with coefficients distributions, i.e.*

$$T = i^{p^2} \sum_{|I|=|J|=p} T_{I,J} dz_I \wedge d\bar{z}_J,$$

where $T_{I,J} \in \mathcal{D}'(\Omega)$.

A current of bidegree (p, p) acts on the space $\mathcal{D}_{q,q}(\Omega)$, $q := n - p$, of test forms of bidegree (q, q) as follows: if

$$\Psi = i^{q^2} \sum_{|K|=|L|=q} \psi_{K,L} dz_K \wedge d\bar{z}_L,$$

where $\psi_{K,L} \in \mathcal{D}(\Omega)$, then

$$< T, \Psi >= \sum_{|I|=p,|J|=q} < T_{I,I'}, \psi_{J,J'} >,$$

since $i^{p^2} dz_I \wedge d\bar{z}_J \wedge i^{q^2} dz_K \wedge d\bar{z}_L = \varepsilon_{I,J}\varepsilon_{K,L} i dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge i dz_n \wedge d\overline{z_n}$.

One defines similarly currents of bidegree (p, q).

REMARK 2.30. *We shall also say that a current of bidegree (p, p) is a current of bidimension $(n - p, n - p)$, since it acts on forms of bidegree $(n - p, n - p)$.*

Recall the decomposition $d = \partial + \overline{\partial}$. We have defined the differential dS of a current, we can similarly define the derivatives ∂S and $\overline{\partial} S$ as follows. If S is a smooth differential form of bidegree (p, p) and $\Psi \in \mathcal{D}_{(n-p-1,n-p)}(\Omega)$, observe that

$$\partial S \wedge \Psi = dS \wedge \Psi = d(S \wedge \Psi) - S \wedge d\Psi = d(S \wedge \Psi) - S \wedge \partial\Psi.$$

hence $\int_\Omega \partial S \wedge \Psi = -\int_\Omega S \wedge \partial\Psi$. This suggests the following:

DEFINITION 2.31. *Let S be a current of bidegree (p, p). The current ∂S is a current of bidegree $(p + 1, p)$ defined by*

$$< \partial S, \Psi >= - < S, \partial\Psi >$$

for all $\Psi \in \mathcal{D}_{(n-p-1,n-p)}(\Omega)$. We define $\bar{\partial} S$ similarly.

We set $d^c := (i/2\pi)(\overline{\partial} - \partial)$. Observe that d and d^c are real differential operators of order one and

$$dd^c = \frac{i}{\pi}\partial\overline{\partial}$$

is a real differential operator of order 2. These operators act naturally on differential forms, their actions are extended to currents by duality.

LEMMA 2.32. *Let S be a current of bidegree (p, p), $0 \leq p \leq n-1$, and Ψ a smooth form of bidegree (q, q), $0 \leq q \leq n-p-1$. If α is a smooth form of bidegree $(n-p-q-1, n-p-q-1)$ then*

$$dS \wedge d^c\Psi \wedge \alpha = d\Psi \wedge d^cS \wedge \alpha$$

and

$$S \wedge dd^c\Psi \wedge \alpha - dd^cS \wedge \Psi \wedge \alpha = d(S \wedge d^c\Psi - \Psi \wedge d^cS) \wedge \alpha.$$

If Ψ is a smooth test form of bidegree $(n-p-1, n-p-1)$ then

$$< dd^cS, \Psi >=< S \wedge dd^c\Psi, 1 > .$$

PROOF. Observe that

$$\begin{aligned} d\Psi \wedge d^cS &= \left(\partial\Psi + \overline{\partial}\Psi\right) \wedge \frac{i}{2\pi}\left(\overline{\partial}S - \partial S\right) \\ &= \frac{i}{2\pi}\left(\partial\Psi \wedge \overline{\partial}S + \partial S \wedge \overline{\partial}\Psi + \overline{\partial}\Psi \wedge \overline{\partial}S - \partial\Psi \wedge \partial S\right). \end{aligned}$$

Similarly

$$dS \wedge d^c\Psi = \frac{i}{2\pi}\left(\partial S \wedge \overline{\partial}\Psi + \partial\Psi \wedge \overline{\partial}S + \overline{\partial}S \wedge \overline{\partial}\Psi - \partial S \wedge \partial\Psi\right),$$

in the weak sense of currents on Ω.

The currents $d\Psi \wedge d^cS$ and $dS \wedge d^c\Psi = -d^c\Psi \wedge dS$ have the same $(p+q+1, p+q+1)-$part hence

$$d\Psi \wedge d^cS \wedge \alpha = dS \wedge d^c\Psi \wedge \alpha,$$

since $\partial S \wedge \partial\Psi \wedge \alpha = 0 = \overline{\partial}\Psi \wedge \overline{\partial}S \wedge \alpha$. On the other hand,

$$d(\Psi \wedge d^cS - S \wedge d^c\psi) = d\Psi \wedge d^cS + \Psi \wedge dd^cS - dS \wedge d^c\Psi - S \wedge dd^c\Psi.$$

The last formula is obtained taking $\alpha = 1$ and applying Lemma 2.28. □

4. Positive currents

The notion of positive current was introduced y Lelong to unify the notion of plurisubharmonic functions and intergation along analytic sets (see [**Lel69**]).

4.1. Positive forms. Let V be a complex vector space of complex dimension $n \geq 1$. Consider a basis $(e_j)_{1\leq j\leq n}$ of V and denote by $(e_j^*)_{1\leq j\leq n}$ the dual basis of V^*. Then any $v \in V$ can be written

$$v = \sum_{1\leq j\leq n} e_j^*(v)e_j.$$

Since we are mainly interested in the case where $V = T_xX$ is the complex tangent space to a complex manifold X of dimension n, we use complex differential notations. A vector $v \in V$ acts as a derivation on germs of smooth functions in a neighborhood of the origin in V by

$$v \cdot f(0) := D_v f(0).$$

If $z = (z_1, \cdots, z_n)$ are complex coordinates identifying V with $\mathbb{C}^n$, then $e_j = \frac{\partial}{\partial z_j}$ is the partial derivative with respect to z_j and $e_j^* = dz_j$.

The exterior algebra of V is

$$\Lambda V_{\mathbb{C}}^* := \oplus \Lambda^{p,q}V^*, \quad \Lambda^{p,q}V^* := \Lambda^p V^* \otimes \Lambda^q \overline{V^*},$$

where $\Lambda^p V^*$ is the complex vector space of alternated $\mathbb{C}-$linear $p-$forms. A complex basis of the space $\Lambda^p V^*$ is given by the $dz_{k_1} \wedge \ldots \wedge dz_{k_p}$ where $K = (k_1, \ldots, k_p)$ vary in the set of ordered multi-indices of length $|K| = p$. Thus $\dim_{\mathbb{C}} \Lambda^p V^* = \begin{pmatrix} n \\ p \end{pmatrix}$.

The complex vector space V has a canonical orientation given by the $(n, n)-$form

$$\beta_n(z) := \frac{i}{2} dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \frac{i}{2} dz_n \wedge d\overline{z}_n = dx_1 \wedge dy_1 \ldots \wedge dx_n \wedge dy_n,$$

where $z_j = x_j + iy_j$, $j = 1, \ldots, n$. If $(w_1, \ldots, w_n)$ is another (complex) coordinate system on V, then $dw_1 \wedge \ldots \wedge dw_n = \det(\frac{\partial w_j}{\partial z_k}) dz_1 \wedge \ldots \wedge dz_n$ so that

$$\beta_n(w) = |\det(\partial w_j / \partial z_k)|^2 \beta_n(z).$$

In particular any complex manifold inherits a canonical orientation induced by its complex structure.

Definition 2.33.

1) A (n, n)-form $\nu \in \Lambda^{n,n}V^*$ is positive if in some local coordinate system $(z_1, \ldots, z_n)$ it can be written $\nu = \lambda(z)\beta_n(z)$, with $\lambda(z) \geq 0$.

2) A (p, p)-form $f \in \Lambda^{p,p}V^* (0 \leq p \leq n)$ is strongly positive if it is a linear combination with positive coefficients of a finite number of decomposable $(p, p)-$forms, i.e. forms of the type

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots i\alpha_p \wedge \overline{\alpha}_p,$$

where $\alpha_1, \cdots, \alpha_p$ are $(1, 0)-$forms on V.

3) A $(p, p)-$form $u \in \Lambda^{p,p}V^* (1 \leq p \leq n-1)$ is (weakly) positive if for all $(1, 0)-$forms $\alpha_j \in \Lambda^{1,0}V^*, (1 \leq j \leq q := n-p)$, the $(n, n)-$form $u \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots i\alpha_q \wedge \overline{\alpha}_q$ is positive.

EXAMPLES 2.34.

1) For all $(1,0)-$*forms* $\gamma_1, \ldots, \gamma_p \in \Lambda^{1,0}V^*$, *the* (p,p)*-form*

$$i\gamma_1 \wedge \overline{\gamma}_1 \wedge \ldots i\gamma_p \wedge \overline{\gamma}_p = i^{p^2}\gamma \wedge \overline{\gamma},$$

is positive, where $\gamma := \gamma_1 \wedge \ldots \wedge \gamma_p$.

2) For all $\alpha \in \Lambda^{p,0}V^*$, *the* $(p,p)-$*form* $i^{p^2}\alpha \wedge \overline{\alpha}$ *is positive, hence any strongly positive* $(p,p)-$*form is (weakly) positive. If* $\alpha \in \Lambda^{p,0}V^*$ *and* $\beta \in \Lambda^{q,0}V^*$, *then*

$$i^{p^2}\alpha \wedge \overline{\alpha} \wedge i^{q^2}\beta \wedge \overline{\beta} = i^{(p+q)^2}\alpha \wedge \beta \wedge \overline{\alpha} \wedge \overline{\beta}.$$

In particular if $p+q=n$, *then* $\alpha \wedge \beta = \lambda dz_1 \wedge \ldots \wedge dz_n$, $\lambda \in \mathbb{C}$, *hence*

$$i^{n^2}\alpha \wedge \beta \wedge \overline{\alpha} \wedge \overline{\beta} = |\lambda|^2\beta_n(z) \geq 0.$$

The following lemma will be useful in the sequel.

LEMMA 2.35. *Let* $(z_1, \ldots, z_n)$ *be a coordinate system in* V. *The complex vector space* $\Lambda^{p,p}V^*$ *is generated by the strongly positive forms*

$$\gamma := i\gamma_1 \wedge \overline{\gamma}_1 \wedge \ldots i\gamma_p \wedge \overline{\gamma}_p, \tag{4.1}$$

where the $(1,0)-$*forms* γ_l *are of the type* $dz_j \pm dz_k$ *or* $dz_j \pm i dz_k$.

PROOF. The proof relies on the following polarization identities for the forms $dz_j \wedge d\bar{z}_k$

$$\begin{aligned} 4dz_j \wedge d\bar{z}_k &= (dz_j + dz_k) \wedge \overline{(dz_j + dz_k)} - (dz_j - dz_k) \wedge \overline{(dz_j - dz_k)} \\ &+ i(dz_j + idz_k) \wedge \overline{(dz_j + idz_k)} - i(dz_j - idz_k) \wedge \overline{(dz_j - idz_k)}. \end{aligned}$$

Since

$$dz_J \wedge d\overline{z}_K = dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_p} = \pm \bigwedge_{1 \leq s \leq p} dz_{j_s} \wedge d\overline{z}_{k_s},$$

it follows that the $(p,p)-$forms γ_s of type (4.1) generate the space $\Lambda^{p,p}V^*$ over $\mathbb{C}$. ☐

COROLLARY 2.36.

1. All positive forms $u \in \Lambda^{p,p}V^*$ *are real i.e.* $\bar{u} = u$ *and if* $u = i^{p^2}\sum_{|I|=|J|=p} u_{I,J} dz_I \wedge d\bar{z}_J$ *then the coefficients satisfy the hermitian symmetry relation* $\bar{u}_{I,J} = u_{J,I}$ *for all* I, J.

2. A form $u \in \Lambda^{p,p}V^*$ *is positive if and only if its restriction to any complex subspace* $W \subset V$ *of dimension* p *is a positive form of top degree on* W.

PROOF. Let (θ_s) be the basis of $\Lambda^{p,p}V^*$ dual to the basis of strongly positive forms (γ_s) of $\Lambda^{n-p,n-p}V^*$ given by Lemma 2.35. Observe that strongly positive forms are real. If $\alpha \in \Lambda^{p,p}V^*$ is positive, we decompose it as $\alpha = \sum_s c_s\theta_s$ with $c_s = \alpha \wedge \gamma_s \geq 0$ for any s. Thus $\overline{\alpha} = \alpha$.

Suppose that $\alpha = i^{p^2}\sum_{|I|=p,|J|=p} \alpha_{I,J} dz_I \wedge d\overline{z}_J$, then

$$\overline{\alpha} = (-1)^{p^2} i^{p^2} \sum_{|I|=p,|J|=p} \overline{\alpha_{I,J}} d\overline{z}_I \wedge dz_J.$$

Since $d\overline{z}_I \wedge dz_J = (-1)^{p^2} dz_J \wedge d\overline{z}_I$, it follows that $\overline{\alpha}_{I,J} = \alpha_{J,I}$, $\forall I, J$.

If $W \subset V$ is a complex subspace of dimension p, there exists a system of complex coordinates $(z_1, \ldots, z_n)$ such that

$$W = \{z_{p+1} = \ldots = z_n = 0\}.$$

Thus $\alpha_{|W} = c_W \frac{i}{2} dz_1 \wedge d\overline{z}_1 \wedge \ldots \frac{i}{2} dz_p \wedge d\overline{z}_p$, where c_W is given by

$$\alpha \wedge \frac{i}{2} dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \frac{i}{2} dz_n \wedge d\overline{z}_n = c_W \beta_n(z).$$

Therefore if α is positive then $\alpha_{|W} \geq 0$ for any complex subspace $W \subset V$ of dimension p. The converse is true since the $(n-p, n-p)$-forms $\wedge_{j>p} i dz_j \wedge d\overline{z}_j$ generate the cone of strongly positive (p,p)-forms when W varies among all complex subspaces W of dimension p. ☐

COROLLARY 2.37. *A $(1,1)$-form $\omega = i \sum_{j,k} \omega_{jk} dz_j \wedge d\bar{z}_k$ is positive if and only if the matrix (ω_{jk}) is a hermitian semi-positive matrix i.e.*

$$\sum_{j,k} \omega_{jk} \xi_j \bar{\xi}_k \geq 0 \quad \textit{for all} \quad \xi \in \mathbb{C}^n.$$

PROOF. Indeed, if $W = \mathbb{C} \cdot \xi$ is a complex line generated by the vector $\xi \neq 0$, then $\omega_{|W} = (\sum_{j,k} \omega_{j,k} \xi_j \overline{\xi_k}) i dt \wedge d\overline{t}$. ☐

REMARK 2.38. *There is a canonical correspondence between hermitian forms and real $(1,1)$-forms on V. Indeed in a system of complex coordinates $(z_1, \ldots, z_n)$, a hermitian form can be written as*

$$h = \sum_{1 \leq j,k \leq n} h_{j,k} dz_j \otimes d\overline{z}_k.$$

The associated $(1,1)-$form

$$\omega_h := \frac{i}{2} \sum_{1 \leq j,k \leq n} h_{j,k} dz_j \wedge d\overline{z}_k$$

is a real $(1,1)$-form on V. This correspondence does not depend on the system of complex coordinates since for all $\xi, \eta \in V$,

$$\omega_h(\xi, \eta) = \frac{i}{2} \sum_{1 \leq j,k \leq n} h_{j,k} (\xi_j \overline{\eta_k} - \eta_j \overline{\xi_k}) = -\Im h(\xi, \eta),$$

and $h_{k,j} = \overline{h_{j,k}}$. Moreover

$$\omega_h(i\xi, \eta) = -\Im h(i\xi, \eta) = \Re h(\xi, \eta),$$

for all $(\xi, \eta) \in V$, which proves that h is entirely determined by ω_h.

Observe finally that h is a positive hermitian form on V if and only if the $(1,1)-$form ω_h is positive.

The notions of positivity (strong and weak) usually differ, but they do coincide in bidegree $(1,1)$:

PROPOSITION 2.39. *A* $(1,1)$*-form is strongly positive if and only if it is weakly positive. In particular if* $\alpha \in \Lambda^{p,p}V^*$ *is a weakly positive form on* V *then for all positive* $(1,1)-$*forms* $\omega_1, \dots \omega_q$ *with* $p+q \leq n$*, the* $(p+q,p+q)$*-form* $\alpha \wedge \omega_1 \wedge \dots \wedge \omega_q$ *is weakly positive.*

PROOF. Let $\omega \in \Lambda^{1,1}V^*$ be a positive $(1,1)$-form on V. Diagonalizing the hermitian form h associated to ω, we see that

$$\omega = \sum_{1 \leq j \leq r} i\gamma_j \wedge \overline{\gamma_j},$$

where $\gamma_j \in V^*$ for $1 \leq j \leq r$. Thus ω is strongly positive. □

We finally define the positivity of differential forms as follows:

DEFINITION 2.40. *A smooth differential* (q,q)*-form* $\phi \in \mathcal{D}_{q,q}(\Omega)$ *in an open set* $\Omega \subset \mathbb{C}^n$ *is positive (resp. strongly positive) if for all* $x \in \Omega$*, the* (q,q)*-form* $\phi(x) \in \Lambda^{q,q}\mathbb{C}^n$ *is positive (resp. strongly positive).*

4.2. Positive currents. The duality between positive and strongly positive forms enables us to define the corresponding positivity notions for currents:

DEFINITION 2.41. *A current* T *of bidimension* (q,q) *is (weakly) positive if* $\langle T, \phi \rangle \geq 0$ *for all strongly positive differential test forms* ϕ *of bidegree* (q,q)*.*

It follows from the definitions that T is positive iff for all $\alpha_1, \cdots, \alpha_q \in \mathcal{D}_{1,0}(\Omega)$, $T \wedge i\alpha_1 \wedge \bar{\alpha_1} \wedge \cdots i\alpha_q \wedge \bar{\alpha_q} \geq 0$ as a distribution on Ω.

Here is an important consequence of this definition.

PROPOSITION 2.42. *Let* $T \in \mathcal{D}'_{q,q}(X)$ *be a positive current and set* $p := n-q$*. Then* T *can be extended as a real current of order* 0 *i.e.*

$$T = i^{p^2} \sum_{|I|=|J|=p} T_{I,J} dz_I \wedge d\overline{z}_J,$$

where the coefficients $T_{I,J}$ *are complex measures in* Ω *satisfying the hermitian symmetry* $\overline{T_{I,J}} = T_{J,I}$ *for any multi-indices* $|I| = |J| = p$*.*

Moreover for any I*,* $T_{I,I} \geq 0$ *is a positive Borel measure in* Ω *and the (local) total variation measure*

$$\|T\| := \sum_{|I|=|J|=p} |T_{I,J}|$$

of the current T *is bounded from above by the trace measure,*

$$\|T\| \leq c_{p,n} \sum_{|K|=p} T_{K,K}, \tag{4.2}$$

where $c_{p,n} > 0$ *are universal constants.*

Proof. Since positive forms are real, it follows by duality that every positive current is real, as the current $\overline{T}$ is defined by the formula $\overline{T}(\phi) := \overline{T(\overline{\phi})}$ for $\phi \in \mathcal{D}_{n-p,n-p}(X)$.

It follows from Lemma 2.35 that any form $\varphi \in \mathcal{D}_{n-p,n-p}(U)$ can be written as $\phi = \sum_s c_s \gamma_s$ where (γ_s) is a basis of strongly positive $(n-p, n-p)-$forms

If ϕ is real the functions c_s are real C^∞-smooth with compact support in Ω. Writing c_s as a difference of non-negative C^∞-smooth functions with compact support, the real form ϕ can be written as the difference of strongly positive forms, hence $T(\phi)$ is a difference of two positive reals. We infer $\overline{T_{I,J}} = T_{J,I}$ for all multi- indices $|I| = |J| = p$.

Observe now that

$$T_{I,I}\beta_n = T \wedge \frac{i^{p^2}}{2^p} dz_{I'} \wedge d\overline{z}_{I'} \geq 0,$$

while the proof of Lemma 2.35 yields

$$T_{I,J} 2^p \beta_n = T \wedge i^{p^2} dz_{I'} \wedge d\overline{z}_{J'} = \sum_{\nu \in \{0,1,2,3\}} \varepsilon_\nu T \wedge \gamma_\nu,$$

where $\varepsilon_\nu = \pm 1, \pm i$ and

$$\gamma_\nu = \bigwedge_{1 \leq s \leq p} i\ell_{\nu,s} \wedge \overline{\ell_{\nu,s}},$$

where $\ell_{\nu,s}$ are $\mathbb{C}-$linear forms on $\mathbb{C}^n$. Since $T \wedge \gamma_a$ is a positive measure on Ω, the distributions $T_{I,J}$ are complex measures in Ω such that

$$2^p |T_{I,J}| \beta_n \leq \sum_\nu T \wedge \gamma_\nu.$$

The only terms that matter here are those for which

$$\gamma_\nu = \bigwedge_{1 \leq s \leq p} i\ell_{\nu,s} \wedge \overline{\ell_{\nu,s}} \neq 0.$$

We can thus assume that the $\mathbb{C}-$linear forms $\ell_{\nu,1}, \ldots, \ell_{\nu,s}$ are linearly independent. Fix such ν and set $\ell_s = \ell_{\nu,s}$. There exists a unitary transformation $A : \mathbb{C}^n_z \longrightarrow \mathbb{C}^n_w$ such that the direct image $A_\star$ sends the subspace of $(\mathbb{C}^n)^*$ generated by the $\mathbb{C}$-linear 1-forms $(\ell_s)_{1 \leq s \leq p}$ onto the subspace generated by the 1-forms $(dw_s)_{1 \leq s \leq p}$. Therefore

$$A_\star(\bigwedge_{1 \leq s \leq p} \ell_s \wedge \overline{\ell}_s) = \bigwedge_{1 \leq s \leq p} A_\star(\ell_s) = |\det(\partial \tilde{\ell}_s / \partial w_k)|^2 \bigwedge_{1 \leq k \leq p} dw_k \wedge d\bar{w}_k.$$

Hence

$$A_\star(T \wedge \gamma_\nu) = a_\nu A_\star(T) \wedge \bigwedge_{1 \leq k \leq p} \frac{i}{2} dw_k \wedge d\bar{w}_k = a_\nu A_\star(T) \frac{i^p}{2^q} dw_K \wedge d\overline{w}_K,$$

where $K := (1, 2, \ldots, p)$ and for all $\nu, a_\nu \geq 0$ is a constant which does not depend on T. Thus

$$A_\star(T \wedge \gamma_\nu) \leq a_\nu A_\star(T) \wedge \beta_n(w).$$

and

$$T \wedge \gamma_\nu \leq a_\nu T \wedge (A^{-1})_\star(\beta_n) = a_\nu T \wedge \beta_p,$$

since β is invariant by unitary transformations on $\mathbb{C}^n$. Observe that

$$\beta_p = \sum_{|K|=p} \left(\frac{i}{2}\right)^p \bigwedge_{1 \leq s \leq p} dz_{k_s} \wedge d\bar{z}_{k_s} = \sum_{|K|=p} \frac{i^{q^2}}{2^p} dz_K \wedge d\bar{z}_K,$$

hence $T \wedge \beta_p = \sum_{|L|} T_{L,L}$ and $T \wedge \gamma_\nu \leq a_\nu \sum_{|L|=p} T_{L,L}$ for all ν. It follows that there exists a uniform constant $c_{p,n} > 0$ such that

$$|T_{I,J}| \leq c_{p,n} \sum_{|L|=p} T_{L,L}.$$

□

COROLLARY 2.43. *Let T be a positive current of bidegree (p,p) and v a continuous strongly positive (m,m)-form, $p+m \leq n$. The current $T \wedge v$ is positive.*

In particular for all continuous positive $(1,1)$-forms $\alpha_1, \ldots, \alpha_m$,

$$T \wedge \alpha_1 \wedge \cdots \wedge \overline{\alpha_m} \geq 0$$

is a positive current.

4.3. Examples.

4.3.1. *Currents of bidegree* $(1,1)$. Let X be a (connected) complex manifold of dimension n. We let $PSH(X)$ denote the convex cone of plurisubharmonic functions in X which are not identically $-\infty$.

PROPOSITION 2.44. *If $u \in PSH(X)$ then the current $T_u := dd^c u$ is a closed positive current of bidegree $(1,1)$ on X.*

PROOF. The property is local so we can assume $X = \Omega$ is an open subset of $\mathbb{C}^n$. Assume first that $u \in PSH(\Omega) \cap C^2(\Omega)$. Then

$$dd^c u = \frac{i}{\pi} \sum_{1 \leq j,k \leq n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} dz_j \wedge \partial \overline{z_k}$$

is a strongly positive $(1,1)$-form since for each $z \in \Omega, \xi \in \mathbb{C}^n$

$$\sum_{1 \leq j,k \leq n} \frac{\partial^2 u(z)}{\partial z_j \partial \overline{z_k}} \xi_j \overline{\xi_k} \geq 0.$$

To treat the general case we regularize u by using radial mollifiers and set $u_\varepsilon := u \star \rho_\varepsilon$. This is a smooth plurisubharmonic function in the open set $\Omega_\varepsilon := \{z \in \Omega; \mathrm{dist}(z; \partial\Omega) > \varepsilon\}$. Since $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, it follows that $dd^c u_\varepsilon$ converges to $dd^c u$ in the weak sense of currents, hence $dd^c u$ is a positive current in Ω. □

Conversely one can show that a closed positive current T of bidegree $(1,1)$ can be locally written $T = dd^c u$, where u is a (local) plurisubharmonic function (see Exercise 2.12). Such a function is called a local potential of T. Observe that two local potentials differ by a pluriharmonic function h, i.e. a smooth function such that $dd^c h = 0$ in Ω.

4.3.2. *Current of integration over a complex analytic set.*

Complex submanifolds. Let $Z \subset X$ be a complex submanifold of X of dimension $m \geq 1$. Its complex structure induces a natural orientation on Z, we can thus integrate a smooth test form ψ of top degree $2m$ on Z. Using a partition of unity we may assume that the support of Ψ lies in a coordinate chart (D, z). Thus $\psi(z) = f(z)\beta_m(z)$ in D, where f is a test function in D and by definition

$$\int_D \psi = \int_D f(z_1, \ldots, z_m) \frac{i}{2} dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \frac{i}{2} dz_m \wedge d\overline{z}_m.$$

This formula does not depend on the local coordinates, as follows from the change of variables formula.

Definition 2.45. *We let $[Z]$ denote the current of integration over Z. It is a positive current of bidimension (m,m) defined by*

$$< [Z], \varphi > := \int_Z j^*(\varphi)$$

for $\varphi \in \mathcal{D}_{m,m}(X)$, where $j : Z \to X$ denotes the embedding of Z in X.

If Φ is a strongly positive test form of bidegree (m,m) then $j^*(\Phi)$ is a positive volume form on Z, hence $\int_Z j^*(\Phi) \geq 0$, which shows that $[Z]$ is a positive current of bidimension (m,m) on X.

If Z if a *closed* complex submanifold of X (no boundary), Stokes formula shows that for all $\Psi \in \mathcal{D}_{2m-1}(X)$,

$$< d[Z], \Psi > = - < [Z], d\psi > = - \int_Z j^*(d\psi) = - \int_Z dj^*(\psi) = 0,$$

thus $[Z]$ is a closed positive current on X.

Analytic subsets. Let Z be a (closed) complex analytic subset of X of pure dimension m, $1 \leq m \leq n$. We refer the reader to [**?**] for basics on analytic sets. One can consider the positive current $[Z]$ of integration over the complex manifold Z_{reg} of regular points of Z i.e. for any test form Φ of bidegree $(n-m, n-m)$ we set

$$\langle [Z], \Phi \rangle := \int_{Z_{reg}} j^* \Phi,$$

where $j : Z \to X$ is the canonical embedding.

It is not obvious that this integral converges since $j^*\Phi$ is not compactly supported in Z_{reg}. This has been proved by Lelong who showed the following remarkable result:

THEOREM 2.46. *The current $[Z]$ is a closed positive current of bidegree (m,m) on X.*

We refer the reader to [**Chir89**, Theorem 14.1] for a proof. In particular if f is a holomorphic function in X which is not identically 0, then its zero locus $Z(f)$ defines a positive closed current on X which satifies the Poincaré-Lelong equation

$$dd^c \log |f| = [Z(f)]$$

in the sense of currents.

4.4. The trace measure of a positive current. Let X be a hermitian manifold; for all $x \in X$ the complex tangent space $T_x X$ is endowed with a positive definite hermitian scalar product $h(x)$ which depends smoothly on x. We let ω denote its fundamental $(1,1)$-form.

DEFINITION 2.47. *Let $T \in \mathcal{D}'_{p,p}$ be a positive current of bidegree (p,p), $1 \leq p \leq n$. The trace measure of T is*

$$\sigma_T := \frac{1}{(n-p)!} T \wedge \omega^{n-p}.$$

In a local coordinates $(U, z_1, \ldots, z_n)$ we can write

$$h := \sum_{j,k} h_{j,k} dz_j \otimes dz_k,$$

where $(h_{j,k})$ is a positive definite hermitian matrix with smooth entries in U. For $x \in U$, the complex cotangent space $T_x^* X$ can also be endowed with a natural hermitian scalar product: applying Hilbert-Schmidt orthonormalization process to the the basis $(dz_1(x), \ldots, dz_n(x))$, we construct an orthonormal basis $(\zeta_1(x), \ldots, \zeta_n(x))$ of $T_x^* X$.

Thus $(\zeta_1, \ldots, \zeta_n)$ is a system of smooth differential $(1,0)$-forms in U such that $(\zeta_1(x), \ldots, \zeta_n(x))$ is an orthonormal basis of $T_x^* X$. We say that $(\zeta_1, \ldots, \zeta_n)$ is a local orthonormal frame (with respect to the hermitian product h) of the cotangent bundle $T^*(X)$ over U. Writing

$$h = \sum_{1 \leq j \leq n} \zeta_j \otimes \overline{\zeta}_j,$$

we get

$$\omega = \frac{i}{2} \sum_{1 \leq j \leq n} \zeta_j \wedge \overline{\zeta_j}$$

and

$$\frac{\omega^q}{q!} = \frac{i^{q^2}}{2^q} \sum_{|K|=q} \zeta_K \wedge \overline{\zeta}_K.$$

Fix $T \in \mathcal{D}'^{+}_{p,p}$ and set $q := n - p$. We can decompose T as

$$T = \sum_{|I|=|J|=q} i^{q^2} T_{I,J} \zeta_I \wedge \overline{\zeta}_J,$$

where $T_{I,J} \in \mathcal{D}(U)$ and $\zeta_I := \zeta_{i_1} \wedge \ldots \wedge \zeta_{i_q}$. A simple computation yields

$$\sigma_T = \left(\sum_{|I|=q} T_{I,I} \right) \bigwedge_{1 \leq j \leq n} \frac{i}{2} \zeta_j \wedge \overline{\zeta_j},$$

i.e. $\sigma_T = \sum_{|I|=q} T_{I,I}$, identifying currents of top degree and distributions. We let the reader check that if X is a domain in $\mathbb{C}^n$ equipped with the standard euclidean metric $\sum_{1 \leq j \leq n} dz_j \otimes \overline{dz_j}$ and if $T \in \mathcal{D'}_{p,p}(X)$, then

$$\sigma_T = \sum_{|I|=q} T_{I,I}.$$

Proposition 2.42 can now be reformulated as follows:

Corollary 2.48. *Let $T \in \mathcal{D'}_{p,p}(X)$ be a positive current. If we decompose T locally, $T = \sum_{|I|=|J|=p} i^{p^2} T_{I,J} dz_I \wedge d\overline{z_J}$, the total variation $\|T\| = \sum_{|I|=|J|=p} |T_{I,J}|$ of T is dominated by the trace measure σ_T,*

$$\sigma_T \leq \|T\| \leq c_{n,p} \sigma_T,$$

where $c_{n,p}$ is an absolute constant independent of T.

In particular the topology of weak convergence in the sense of distributions coincides, for positive currents, with the weak convergence in the sense of Radon measures.

Example 2.49. *Let $u \in PSH(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is a domain. The trace measure of the closed positive current $T := dd^c u$ coincides with the Riesz measure of u, i.e.*

$$\sigma_T := dd^c u \wedge \beta_{n-1} = \frac{1}{2\pi} \Delta u$$

This formula can be generalized to any plurisubharmonic function on a complex hermitian manifold (X, ω), replacing β by ω and Δ by Δ_ω the Laplace operator associated to the hermitian metric ω.

5. Exercises

Exercise 2.1. *Let φ be a subharmonic function in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, i.e. an upper semi-continuous function which is locally integrable and satisfies*

$$\Delta\varphi := \frac{1}{4} \sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_i} \geq 0$$

in the sense of distributions. Show that φ is pluri-subharmonic if and only if for all $A \in GL(n, \mathbb{C})$,

$$\varphi_A : z \in \mathbb{C}^n \mapsto \varphi(A \cdot z) \in \mathbb{R}$$

is subharmonic. (see [**Hor94**]*)*

EXERCISE 2.2. *Show that a convex function $f : \mathbb{R} \to \mathbb{R}$ which is bounded from above is constant. Use this to prove that a plurisubharmonic function $\varphi : \mathbb{C}^n \to \mathbb{R}$ which is bounded from above is constant.*

EXERCISE 2.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $\varphi : \Omega \to \mathbb{R}$.*

1) Show that φ is pluriharmonic if and only if it is locally the real part of a holomorphic function.

2) Show that φ is pluharmonic iff φ and $-\varphi$ are plurisubharmonic in Ω.

3) Show that if φ is plurisubharmonic in Ω and harmonic in the real sense then φ is pluriharmonic in Ω.

EXERCISE 2.4.

1) Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function. Show that

$$\varphi_{|\mathbb{R}^n} \in L^1_{loc}(\mathbb{R}^n).$$

2) Let φ_j be a sequence of plurisubharmonic functions in $\mathbb{C}^n$ such that $\varphi_j \to \varphi$ in $L^1_{loc}(\mathbb{C}^n)$. Show that

$$\varphi_{j_{|\mathbb{R}^n}} \longrightarrow \varphi_{|\mathbb{R}^n} \text{ in } L^1_{loc}(\mathbb{R}^n).$$

(See [**Hor94**]*).*

EXERCISE 2.5. *What is the limit of $\varphi_j(z) = j \log ||z||$ in $\mathbb{C}^n$? Is it in contradiction with the compacity criteria we have established ?*

EXERCISE 2.6. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $F = \{u = -\infty\}$ a closed complete pluripolar set (the $-\infty$ locus of a plurisubharmonic function u). Let φ be a plurisubharmonic function in $\Omega \setminus F$ which is locally bounded near F. Show that φ uniquely extends through F as a plurisubharmonic function.*

EXERCISE 2.7. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $A \subset \mathbb{C}^n$ an analytic subset of complex codimension ≥ 2. Let φ be a plurisubharmonic function in $\Omega \setminus A$. Show that φ uniquely extends through A as a plurisubharmonic function (see* [**Chir89**] *for some help).*

EXERCISE 2.8. *Let $f : \Omega \to \Omega'$ be a proper surjective holomorphic map between two domains $\Omega \subset \mathbb{C}^n$, $\Omega' \subset \mathbb{C}^k$. Let u be a plurisubharmonic function in Ω and set, for $z' \in \Omega'$,*

$$v(z') := \max\{u(z) \ , \ f(z) = z'\}.$$

Show that v is plurisubharmonic in Ω'.

EXERCISE 2.9. *Let u be a plurisubharmonic function in $\mathbb{C}^n$.*

1) Show that for any ball B, $\sup_B u = \sup_{\overline{B}} u$.

2) Generalize 1) to bounded open sets Ω with smooth boundary.

3) Deduce that $K = \overline{\Omega}$ is a regular set: if u_j is a sequence of plurisubharmonic functions which converge to u in L^1_{loc}, then

$$\sup_K u_j \to \sup_K u.$$

4) Using the Riemann mapping theorem, show that a connected compact set $K \subset \mathbb{C}$ is regular.

EXERCISE 2.10. *Let $(u_j) \in SH(\mathbb{R}^k)^{\mathbb{N}}$ be a sequence of subharmonic functions converging to u in L^1_{loc}. Using the linearity of the Laplace operator, show that*

$$Du_j \to Du \text{ in } L^q_{loc} \text{ for all } q < k/(k-1).$$

EXERCISE 2.11. *Let $\Omega \subset \mathbb{C}^n$ be a domain. For $K \subset \Omega$ we let*

$$\hat{K} := \{z \in \Omega \,|\, u(z) \leq \sup_K u, \; \forall u \in PSH(\Omega)\}$$

denote the plurisubharmonic hull of K. Say that Ω is pseudoconvex if $\hat{K}$ is relatively compact in Ω whenever K is.

1) Describe $\hat{K}$ when $n = 1$ and show that any Ω is pseudoconvex.

2) Show that $\Omega := \{z \in \mathbb{C}^n \,|\, 1 < ||z|| < 2\}$ is not pseudoconvex.

3) Show that Ω is pseudoconvex iff

$$z \in \Omega \mapsto -\log \mathrm{dist}(\mathrm{z}, \partial\Omega) \in \mathbb{R}$$

is plurisubharmonic . (See [**Hor90**]*).*

EXERCISE 2.12. *Let T a closed positive $(1,1)$-current on Ω. Show that for any ball $\mathbb{B} \subset \Omega$ there exists a plurisubharmonic function ρ in B such that $dd^c \rho_B = T$ weakly on B. The function ρ is called a local potential of T. Show that local potentials are unique up to addition of a pluriharmonic function.*

CHAPTER 3

The complex Monge-Ampère operator

1. Introduction

Let u is $\mathcal{C}^2$-smooth and plurisubharmonic function in an open set $\Omega \subset \mathbb{C}^n$. Recall that

$$dd^c u = \frac{i}{\pi} \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

is a smooth semi-positive $(1,1)$-form on Ω. This means that the complex Hessian of φ at each point $z \in \Omega$ is a semi-positive hermitian matrix . A simple computation shows that the wedge product $(dd^c u)^n := dd^c u \wedge \cdots \wedge dd^c u$ (n times) is a top degree differential (n,n)-form on Ω, given by the following formula:

$$(dd^c u)^n = \det \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) \beta^n$$

where $\beta := dd^c |z|^2$ is (up to a constant) the standard Kähler metric on $\mathbb{C}^n$ and β^n is (up to a consatnt) the euclidean volume form on $\mathbb{C}^n$. Observe that the right hand side is then a continuous non negative $(n,n)-$form which we identify to a positive measure with density on Ω.

The above formula still makes sense almost everywhere in Ω for plurisubharmonic functions φ which belong to the Sobolev space $W^{2,n}_{loc}(\Omega)$. The right hand side is then an (n,n)-form on Ω with non negative$L^1_{loc}(\Omega)$ coefficient, which will be identifed to a positive Borel measure with L^1_{loc}-density on Ω. It is called the complex Monge-Ampère measure of φ, sometimes denoted $MA(\varphi)$. It is however crucial for applications to consider plurisubharmonic functions which are far less regular.

The complex Monge-Ampère operator is the non linear operator

$$u \mapsto (dd^c u)^n$$

which associates, to a given plurisubharmonic function in some class $DMA(\Omega)$ of plurisubharmonic functions in a given domain Ω, a non-negative Radon measure on Ω, called the Monge-Ampère measure of u.

Observe that when $n := 1$, then

$$dd^c u = \frac{\partial^2 u}{\partial z \partial \bar{z}} dz \wedge d\bar{z} = \frac{\Delta u}{2\pi} dx \wedge dy,$$

is the Riesz measure of u and it is well defined for any $u \in SH(\Omega)$.

In this chapter we explain how to define and study the complex Monge-Ampère operator acting on plurisubharmonic functions which are locally bounded following the pioneering work of Bedford and Taylor [**BT82**]. We show that it is continuous along monotone sequences, but it is not continuous for the L^1_{loc}-convergence when $n \geq 2$.

The terminology comes from the analogy with the real case. For a C^2-smooth convex function u in a convex domains $D \subset \mathbb{R}^N$, the real Monge-Ampère operator is defined for functions by the formula:

$$MA_{\mathbb{R}}(u) = \det\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)$$

wher the right hand is a continuous function in D.

It can be shown that the real Monge-Ampère operator can be extended to the class of all convex functions in a given convex domain (see [**Gut01**]. This is not surprising since convex functions are continuous and even locally Lipschitz in their domain. Moreover, by a deep theorem of Alexandrof, convex functions are twice differentiable almost everywhere (see [**Ale39**]).

In their first seminal work [**BT76**], E.Bedford and B.A.Taylor were able to extend the definition of the complex Monge-Ampère operator to the class of locally bounded plurisubharmonic functions using the notion of closed positive current.

2. The case of continuous plurisubharmonic functions

Here we show that it is not difficult to extend the complex Monge-Ampère operator to the class of *continuous plurisubharmonic* functions on Ω.

Indeed let u be a continuous psh function in Ω and $u_j := u \star \chi_j$ its regularisation by convolution against a radial approximation of the Dirac unit mass at the origin. Let us prove that the sequence of (smooth) measures $(dd^c u_j)^n$ converges weakly in the sense of distributions on Ω. Since u is continuous, by Dini's lemma, the sequence (u_j) decreases to u, locally uniformly in Ω.

We need the following fundamental estimate.

LEMMA 3.1. *Let $u_1, \cdot, u_n$ be smooth plurisubharmonic functions in Ω. Then for any compact sets K, E $K \subset E^\circ \subset E \subset \Omega$, there exists a constant $C = C(K, L) > 0$ such that*

$$\int_K dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq C \|u_1\|_E \cdots \|u_n\|_E,$$

where $\|.\|_E$ is the uniform norm on E.

This estimate can be proved using Stokes formula (see (2.4) below).

PROPOSITION 3.2. *Let u be a continuous plurisubharmonic function in Ω. Then there exists a unique Radon measure μ_u on Ω such that for any decreasing sequence (u_j) of smooth plurisubharmonic functions converging to u pointwise in Ω, the sequence of Radon measures $(dd^c u_j)^n$ converges to μ weakly on Ω.*

PROOF. Let us prove the existence. Let (u_j) be a the sequence obtained by convolution with a radial approximation of the Dirac measure. Each u_j is a smooth plurisubharmonic function in a domain Ω_j and the sequence of domains Ω_j increases to Ω. Since the sequence (u_j) is locally uniformly bounded in Ω, it follows from Lemma 3.1 that the sequence of measures $(dd^c u_j)^n$ has locally uniformly bounded mass on Ω. Therefore it is enough to prove the weak convergence of the sequence of distributions $(dd^c u_j)^n$ against any smooth test function in $\mathfrak{D}(\Omega)$.

This will be a consequence of the following observation. The problem of convergence being local, it is enough to consider a test function with compact support in a small ball $B \Subset \Omega$. Fix such a smooth test function χ with compact support in B. Then for any C^2-smooth psh functions φ and ψ in Ω, we have by Stokes formula,

$$\int_B \chi((dd^c\varphi)^n - (dd^c\psi)^n) = \int_B (\varphi - \psi) dd^c\chi \wedge T,$$

where $T := \sum_{j=0}^{n-1} (dd^c\varphi)^{n-1-j} \wedge (dd^c\psi)^j)$ is a closed positive current on Ω. Since χ is smooth of compact support, it is possible to write it as $\chi = w_1 - w_2$, where w_1, w_2 are smooth psh functions in Ω. Take $w_1 := A|z|^2 + \chi$, where $A > 0$ is big enough so that w_1 is plsurisubharmonic in Ω, and $w_2 := A|z|^2$.

Then if we set $w := w_1 + w_2$, we obtain

$$\left| \int_B \chi \left((dd^c\varphi)^n - (dd^c\psi)^n\right) \right| \leq \int_B |\varphi - \psi| dd^c w \wedge T. \tag{2.1}$$

Therefore if D is an open neighborhood of $\overline{B}$ such that $B \Subset D \Subset \Omega$, then by Chern-Levine-Nirenberg inequality, there exists a uniform constant $C > 0$, depending only on D, Ω, a uniform bound of the second derivatives of χ such that

$$\left| \int_B \chi \left((dd^c\varphi)^n - (dd^c\psi)^n\right) \right| \leq C M(\varphi, \psi) \|\varphi - \psi\|_{\bar{D}}, \tag{2.2}$$

where

$$M(\varphi, \psi) := \sum_{j=0}^{n-1} \|\varphi\|_{\bar{D}}^j \|\psi\|_{\bar{D}}^{n-1-j}.$$

Now by Dini's lemma, (u_j) convergences uniformly to u on $\bar{D}$. Since the sequence (u_j) is uniformly bounded in $\bar{D}$, It follows from (2.4) that the sequence of measures $(dd^c u_j)^n$for $j > j_D$ is a Cauchy sequence of

Radon measures on D. Then it converges to a positive Radon measure on D. It is easy to see that taking an exhaution of Ω by an increasing sequence of domains D_j, all the measures $\mu_{D_j}(u)$ glue into a unique measure on Ω, denoted by μ.

To prove uniqueness, we use again (2.4) to see that the limit does not depend on the approximating sequence (u_j) which converges to u locally uniformly in Ω. This limit $\mu(u)$ is defined to be the Monge-Ampère measure of u on Ω. It is denoted by $((dd^c u)^n$ and called the complex Monge-Ampère measure of u. □

It turns out that the hypothesis of continuity on the psh function u is a strong condition. Indeed it is not preserved by standard constructions as upper envelopes, regularized limsup of psh functions which arise naturally when dealing with the Dirichlet problem for the complex Monge-Ampère operator. Therefore it is desirable to define the complex Monge-Ampère operator for non continuous psh functions, say e.g. for bounded psh functions.

REMARK 3.3. *As one may see from the previous reasoning, it is not clear how to define the complex Monge-Ampère measure of u by approximating u by a decreasing sequence of smooth psh functions, since the convergence in not locally uniform anymore. However by (2.4), we have*

$$\left|\int_B \chi\left((dd^c u_j)^n - (dd^c u_k)^n\right)\right| \leq \int_B |u_j - u_k| d\mu_{j,k}, \tag{2.3}$$

where

$$\mu_{j,k} := dd^c w \wedge \sum_{j=0}^{n-1} (dd^c \varphi)^{n-1-j} \wedge (dd^c \psi)^j).$$

By Lusin Theorem, the function u is μ-almost continuous in Ω for any Borel measure μ on Ω. This means that for any $\varepsilon >$ there exists a compact subset $E \subset D$ such that $\mu(/D \setminus E) < \varepsilon$ and $u \mid E$ is continuous.

We want to apply this observation to each measure $\mu_{j,k}$, but then the compact set E may depend on j,k and the argument above do not work anymore. However we will show that

Claim: *For any $\varepsilon > 0$, there exits a compact subset $E \subset \Omega$ such that $\sup_{j,k} \mu(\mathbb{D} \subset E) < \varepsilon$ and $u \mid E$ is continuous.*

This is one of the main results in pluripotential theory says that plurisubharmonic functions are actually quasi-continuous ([**BT82**]*). This will be proved in section 5.2. and then if u is a bounded psh function in Ω, the convergence of the approximating sequence (u_j) to u is locally quasi-uniform in Ω.*

Let us prove assuming the claim that the sequence of measures $(dd^c u_j)$ converges weakly to $(dd^c u)^n$. Indeed let $\varepsilon >$, then the sequence

(u_j) converges uniformly on E. By (2.3), we have for any j,k

$$\left|\int_B \chi\left((dd^c u_j)^n - (dd^c u_k)^n\right)\right| \tag{2.4}$$

$$\leq \int_{B\cap E} d\mu_{j,k} + \int_{B\setminus E} |u_j - u_k| d\mu_{j,k} \tag{2.5}$$

$$\leq \|u_j - u_k\|_E \mu_{j,k}(B) + 2M \sup_{j,k} \mu(\mathbb{D}\setminus E) \tag{2.6}$$

$$\leq M^n \|u_j - u_k\|_E + 2M\varepsilon, \tag{2.7}$$

where $M > 0$ is a uniform bound of u_j on $\bar{D}$.

This proves the weak convergence of the Radon measures. .

It turns out that proving quasi-continuity is quite involved and we need anyway to define the complex Monge-Ampère operator in another way (see [**BT82**]*).*

Actually to pass from continuous to bounded psh functions is one of the main difficulties when dealing with the complex Monge-Ampère operator in contrast to the real Monge-Ampère operator which deals with convex functions which are continuous.

The main observation is the following. Let T be a closed positive current of bidegree (k,k) $(1 \leq k \leq n-1)$ and u a locally bounded plurisubharmonic function in Ω. It is well known that T can be extended as a differential form with complex Borel measure coefficients on Ω. Then the current uT is well defined by duality, since u is a locally bounded Borel function and hence locally integrable with respect to all the coefficients of T. Therefore we can define the current $dd^c(uT)$ in the weak sense. Now the following simple observation is crucial: the current $dd^c(uT)$ is again a closed positive current on Ω. Indeed, since the problem is local we can assume that the regularizing sequence $u_j \searrow u$ in Ω. Then $u_j T \rightharpoonup uT$ in the weak sense of measures in Ω and by continuity of the operator dd^c for the weak topology, we conclude that $dd^c(u_j T) \rightharpoonup dd^c(uT)$ weakly in the sense of currents in Ω. Now since u_j is smooth, we have by Stokes formula for currents that $dd^c(u_j T) = dd^c u_j \wedge T$ is a positive closed currents. Therefore $dd^c(uT)$ is also a closed positive current in Ω, which will be denoted by $dd^c u \wedge T$ (see [**?**]).

It is now clear that we can repeat this construction: if $u_1, \cdots, u_k$ are locally bounded psh functions, it is possible to define by induction the current $dd^c u_1 \wedge \cdots \wedge dd^c u_k$ by the formula

$$dd^c u_1 \wedge \cdots \wedge dd^c u_k := dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_k),$$

weakly in the sense of currents in Ω, the resulting current being a closed positive current in Ω.

In particular if u is a locally bounded psh function in Ω, then the current of bidegree (n,n) given by $(dd^c u)^n = dd^c u_1 \wedge \cdots \wedge dd^c u_n$, where

$u_1 = \cdots = u_n = u$ can be identified to a positive Borel measure denoted by $(dd^c u)^n$ called the Monge-Ampère measure of u.

Likely this definition coincides with the previous one when u is a continuous psh function. Using ingenious integration by parts and local approximations, Bedford and Taylor proved the following important convergence theorem ([**BT76**]).

THEOREM 3.4. *Let (u_j) and (v_j) be decreasing sequences of locally bounded psh functions in Ω converging to locally bounded psh functions u and v respectively in Ω. Then the sequence of measures $u_j(dd^c v_j)^n$ converges to the measure $u(dd^c v)^n$ weakly in the sense of measures in Ω. The same weak convergence still holds if (u_j) or (v_j) increases almost everywhere in Ω to u or v respectively.*

2.1. Currents of Monge-Ampère type.

2.2. Definitions. Let T be a closed positive (p,p)-current in a domain $\Omega \subset \mathbb{C}^n$, $0 \leq p \leq n-1$. It can be decomposed as

$$T = i^{p^2} \sum_{|I|=p, |J|=p} T_{I,J} dz_I \wedge d\overline{z}_J,$$

where the coefficients $T_{I,J}$ are complex Borel measures.

A locally bounded Borel function u is locally integrable with respect to the coefficients of T hence uT is a well defined current of order 0, setting

$$< uT, \Psi >=< T, u\Psi >,$$

Ψ a continuous form of bidegree $(n-p, n-p)$ with compact support.

If u is a smooth function the current $dd^c u \wedge T$ is a $(p+1, p+1)$-current in Ω defined by

$$< dd^c u \wedge T, \Psi > := < T, dd^c u \wedge \Psi >$$

for $\Psi \in \mathcal{D}_{q,q}(\Omega)$, $q = n - p - 1$. Observe that $\Theta = d^c u \wedge \Psi - u d^c \Psi$ is a smooth form with compact support such that $dd^c u \wedge \Psi - u dd^c \Psi = d\Theta$. Since $dT = 0$ we infer

$$\begin{aligned} < T, dd^c u \wedge \Psi > &= < T, u dd^c \Psi > + < T, d\Theta > \\ &= < T, u dd^c \Psi > = < dd^c(uT), \Psi > . \end{aligned}$$

This motivates the following:

DEFINITION 3.5. *Let T be a closed positive current of bidegree (p,p) in a domain $\Omega \subset \mathbb{C}^n$ and $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. The current $dd^c u \wedge T$ is a $(p+1, p+1)$-current defined by*

$$dd^c u \wedge T := dd^c(uT),$$

i.e.

$$< dd^c u \wedge T, \Psi >=< uT, dd^c \Psi >,$$

for all test forms Ψ of bidegree (q,q), $q := n - p - 1$.

We also need to consider currents of the type $du \wedge d^c u \wedge T$, where $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. Recall that du is locally in $L^p_{loc}(\Omega)$ (with respect to the Lebesgue measure) for any $p < 2$. When u is locally bounded du actually belongs to $L^2_{loc}(\Omega)$ as we now explain.

Since u is (locally) bounded from below, we can add a large constant and assume that $u \geq 0$. It follows that u^2 is plurisubharmonic hence $dd^c u^2 \wedge T$ is a well defined closed positive current if u is smooth, with

$$dd^c u^2 = 2u dd^c u + 2du \wedge d^c u.$$

This motivates the following:

Definition 3.6. *Let $u \in PSH(\Omega) \cap L^\infty(\Omega)$. We set*

$$du \wedge d^c u \wedge T := \frac{1}{2} dd^c (u-m)^2 \wedge T - (u-m) dd^c u \wedge T,$$

where m is a lower bound for u in Ω.

This is a well defined closed current in Ω which does not depend on m (note that $du = d(u-m)$). These currents (a priori of order 2) are of order zero, as they are positive:

Proposition 3.7. *Let (u_j) be locally bounded plurisubharmonic functions in Ω which decrease to $u \in PSH(\Omega) \cap L^\infty_{loc}$. Then*

$$dd^c u_j \wedge T \to dd^c u \wedge T, \quad du_j \wedge d^c u_j \wedge T \to du \wedge d^c u \wedge T,$$

in the weak sense of currents on Ω.

In particular $dd^c u \wedge T$ and $du \wedge d^c u \wedge T$ are apositive currents on Ω, hence their coefficients extend as complex Borel measures on Ω.

Proof. By definition $dd^c u \wedge T = dd^c(uT)$ is a closed current. By assumption the sequence (u_j) converges in $L^1_{loc}(\Omega, \sigma_T)$ hence $u_j T \to uT$ in the weak sense of currents in Ω. The operator dd^c is continuous for the weak convergence of currents hence $dd^c u_j \wedge T \to dd^c u \wedge T$ in the sense of currents in Ω.

The second statement follows from the first one and the fact that $u_j dd^c u_j \wedge T$ converges to $u dd^c u \wedge T$. This latter convergence relies on several technical results that we establish below: it follows from Chern-Levine-Nirenberg inequalities (Theorem 3.13) that the currents $u_j dd^c u_j \wedge T$ have uniformly bounded masses. We can thus consider a cluster point σ. The reader will check in Exercise 3.1 that

$$\sigma \leq u dd^c u \wedge T,$$

since (u_j) is decreasing and $dd^c u_j \wedge T \to dd^c u \wedge T$. To prove the reverse inequality we can use the localization principle (Proposition 3.10) to insure that u_j and u coincide near the boundary of Ω and then use Proposition 3.11 to show that the currents σ and $u dd^c u \wedge T$ have the same total mass.

It remains to prove the positivity property. Since this is a local property we can reduce to the case where u is plurisubharmonic in a neighborhood of $\overline{\Omega}$ and $0 \leq u \leq M$ on Ω. We can then approximate u by a decreasing family of smooth plurisubharmonic functions, $u_\varepsilon := u \star \rho_\varepsilon$, using standard mollifiers.

We know that $dd^c u_\varepsilon \wedge T \to dd^c u \wedge T$ and $du_\varepsilon \wedge d^c u_\varepsilon \wedge T \to du \wedge d^c u \wedge T$ in the sense of currents in Ω. Since $dd^c u_\varepsilon$ is a positive form of bidegree $(1,1)$, we infer that $dd^c u_\varepsilon \wedge T$ and $du_\varepsilon \wedge d^c u_\varepsilon \wedge T$ are positive currents in Ω, hence so are their limits. □

Iterating this process we define by induction the intersection of currents of the above type.

Definition 3.8. *If $u_1, \ldots, u_k \in PSH(X) \cap L^\infty_{\mathrm{loc}}(X)$, and T is a closed positive current of bidimension (m,m), we define the current $dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge T$ by*

$$dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge T := dd^c \left(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_k \wedge T \right).$$

We define similarly $du_1 \wedge d^c u_1 \wedge \ldots \wedge du_k \wedge d^c u_k \wedge T$.

It turns out that these definitions are symmetric in the u_j's, as we shall soon show.

In particular if $V \in PSH(\Omega)$ and $u_1, \ldots, u_k \in PSH(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, the current $dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge dd^c V$ is a well defined closed positive current on Ω.

Definition 3.9. *The complex Monge-Ampère measure of a locally bounded plurisubharmonic function is*

$$(dd^c u)^n := dd^c u \wedge \cdots \wedge dd^c u.$$

It remains to make sure that this is a *good* definition as in the continuous case.

2.3. Localization principle. A technical point that we are going to use on several occasions is that we can arbitrarily modify a bounded plurisubharmonic function near the boundary of a pseudoconvex domain without changing it on a given compact subset.

Fix $\Omega = \{\rho < 0\} \Subset \mathbb{C}^n$ a bounded strictly pseudoconvex domain, with ρ smooth and strictly plurisubharmonic in a neighborhood of $\overline{\Omega}$.

Proposition 3.10. *Fix $K \subset \Omega$ a compact set and $M > 0$. There exists $C > 0$ depending only on K and Ω, a compact subset $E \subset \Omega$ such that $K \subset E^\circ$ and for any $u \in PSH(\Omega) \cap L^\infty(\Omega)$ with $u < 0$ in Ω, there exists $A > 0$ and a bounded plurisubharmonic function $\tilde{u}$ in a neighborhood of $\bar{\Omega}$ such that*

(i) $\tilde{u} = u$ in a neighborhood of K,

(ii) $\tilde{u} = A\rho$ in $\Omega \setminus E$, with $A \leq C \|u\|_{L^\infty(\Omega)}$,

(iii) $u \leq \tilde{u} \leq A\rho$ on Ω.

In particular $\|\tilde{u}\|_{L^\infty(D)} \leq C \|u\|_{L^\infty(\Omega)}$.

PROOF. Consider, for $c > 0$, $D_c = \{z \in \Omega; \rho(z) < -c\}$, and choose $a > 0$ such that $K \subset D_a$. Set $M := \|u\|_{L^\infty(\Omega)}$ and $A := M/a$ so that

$$u \geq A\rho \text{ on } \partial D_a.$$

Pick $b > 0$ so small so that $a < b$ and $A\rho \geq u$ on ∂D_b. The gluing lemma for plurisubharmonic functions shows that the function

$$\tilde{u}(z) = \begin{cases} u(z) & \text{for} \quad z \in D_a, \\ \max\{u(z), A\rho(z)\} & \text{for} \quad z \in D_b \setminus D_a, \\ A\rho(z) & \text{for} \quad z \in \Omega \setminus D_b, \end{cases}$$

is plurisubharmonic in Ω and satisfies all our requirements with $E := \overline{D}_b$ and $C := \max\{\max_{\bar{\Omega}} |\rho|/a, 1\}$. □

We now establish an integration by parts formula due to U. Cegrell [**Ceg04**]:

PROPOSITION 3.11. *Let T be a closed positive current of bidimension $(1,1)$ in Ω. Let $u, v \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ be such that $u, v \leq 0$, $\lim_{z \to \partial\Omega} u(z) = 0$ and $\int_\Omega dd^c v \wedge T < +\infty$. Then*

$$\int_\Omega v dd^c u \wedge T \leq \int_\Omega u dd^c v \wedge T,$$

where the inequality holds in $[-\infty, 0[$. If $\lim_{z \to \partial\Omega} v(z) = 0$, then

$$\int_\Omega v dd^c u \wedge T = \int_\Omega u dd^c v \wedge T, \tag{2.8}$$

provided that $\int_\Omega dd^c u \wedge T < +\infty$.

PROOF. For $\varepsilon > 0$ set $u_\varepsilon := \sup\{u, -\varepsilon\}$ and observe that u_ε is plurisubharmonic in Ω and increases to 0 as ε decreases to 0. The monotone convergence theorem yields

$$\int_\Omega u dd^c v \wedge T = \lim_{\varepsilon \to 0} \int_\Omega (u - u_\varepsilon) dd^c v \wedge T.$$

Set $\Omega_\varepsilon := \{u < -\varepsilon\}$. Then $K := \overline{\Omega}_\varepsilon \subset \Omega$ is a compact subset such that $u_\varepsilon = u$ on $\Omega \setminus K$. Let $D_1 \Subset \Omega$ be a domain close to Ω such that $K \subset D_1$. Let $(\rho_\eta)_{\eta > 0}$ be standard mollifiers. For $\eta > 0$ small enough, the smooth function $(u - u_\varepsilon) \star \rho_\eta$ has compact support contained in the $\eta-$neighborhood $K_\eta \subset \Omega$ of K and converges to $u - u_\varepsilon$ on Ω as η decreases to 0. It follows from Lebesgue's convergence theorem that

$$\int_{D_1} (u - u_\varepsilon) dd^c v \wedge T = \lim_{\eta \to 0} \int_{D_1} (u - u_\varepsilon) \star \rho_\eta dd^c v \wedge T.$$

Since $(u - u_\varepsilon) \star \rho_\eta$ is smooth and has a compact support in D_1 we infer

$$\int_{D_1} (u - u_\varepsilon) \star \rho_\eta dd^c v \wedge T = \int_{D_1} v dd^c((u - u_\varepsilon) \star \rho_\eta) \wedge T \geq \int_{D_1} v dd^c(u \star \rho_\eta) \wedge T.$$

Assume first that v is continuous in a neighborhood of $\bar{D}_1$ and observe that we can choose D_1 so that the positive Borel measure $(-v)dd^c u \wedge T$ has no mass on ∂D_1. Thus

$$(-v)dd^c(u \star \rho_\eta) \wedge T \to (-v)dd^c u \wedge T$$

weakly in the sense of Radon measures in Ω. Therefore

$$\lim_{\eta \to 0} \int_{D_1} v \, dd^c(u \star \rho_\eta) \wedge T = \int_{D_1} v \, dd^c u \wedge T.$$

We infer

$$\int_{D_1} u \, dd^c v \wedge T \geq \int_{D_1} v \, dd^c u \wedge T - \varepsilon \int_{D_1} dd^c v \wedge T. \tag{2.9}$$

If v is not lower semi-continuous we take a decreasing sequence (v_j) of negative continuous plurisubharmonic functions in Ω which converges to v in a neighborhood of $\overline{D}_1$ and apply the last inequality to each function v_j. Choose a domain D_2 such that $D_2 \Subset D_1 \Subset \Omega$. By upper semi-continuity on the compact set $L := \overline{D_2} \subset D_1$, it follows from (2.9) that

$$\begin{aligned}
\int_L u dd^c v \wedge T &\geq \limsup_j \int_L u dd^c v_j \wedge T \\
&\geq \lim_j \int_{D_1} v_j dd^c u \wedge T - \varepsilon \liminf_j \int_{D_1} dd^c v_j \wedge T \\
&\geq \int_{D_1} v dd^c u \wedge T - \varepsilon \int_{\bar{D}_1} dd^c v \wedge T.
\end{aligned}$$

Letting D_1 and D_2 increase to Ω and taking the limit as $\varepsilon \to 0$, we obtain the required inequality provided that $\int_\Omega dd^c v \wedge T < +\infty$. □

Simple examples show that the total mass of the current $dd^c u \wedge T$ in Ω need not be finite:

EXAMPLE 3.12. *Fix $0 < \alpha < 1$ and consider*

$$z \in \mathbb{D} \mapsto u(z) := -(1 - |z|^2)^\alpha \in \mathbb{R}$$

This is a smooth and bounded subharmonic function in the unit disc $\mathbb{D}$, which extends as a Hölder continuous function up to the boundary. The reader can check (Exercise 3.3) that the total mass of Δu is infinite,

$$\int_{\mathbb{D}} dd^c u = +\infty.$$

3. The complex Monge-Ampère measure

3.1. Chern-Levine-Nirenberg inequalities. The following inequalities are due to Chern-Levine-Nirenberg [**CLN69**]. They are the first step towards defining the complex Monge-Ampère operator on bounded plurisubharmonic functions:

THEOREM 3.13. *Let T be a closed positive current of bidimension (k,k) in Ω and $u_1,\ldots,u_k \in PSH(\Omega)\cap L^\infty_{loc}(\Omega)$. Then for all open subsets $\Omega_1 \Subset \Omega_2 \Subset \Omega$, there exists a constant $C = C_{\Omega_1,\Omega_2} > 0$ such that for any compact subsets $K \subset \Omega_1$*

$$\int_{\Omega_1} dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge T \le C\|u_1\|_E \ldots \|u_k\|_E \|T\|_E, \tag{3.1}$$

and

$$\int_{\Omega_1} du_1 \wedge d^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k \wedge T \le C\|u_1\|_E^2 \|u_2\|_E \ldots \|u_k\|_E \|T\|_E, \tag{3.2}$$

where $E := (\Omega_2 \setminus \Omega_1) \cap Supp(T)$.

Here (and in the sequel) we let $\|u\|_E$ denote the L^∞-norm of the function u on the Borel set E.

PROOF. It is enough to prove inequality (3.1) for $k = 1$ and then use induction on k. Set $u = u_1$. We can always assume that $u \le 0$ on Ω_2, since the function $v := u - \sup_{\Omega_2\setminus\Omega_1} u$ is negative on Ω_2 and satisfies $dd^c v = dd^c u$ and $\|v\|_{\Omega_2\setminus\Omega_1} \le 2\|u\|_{\Omega_2\setminus\Omega_1}$.

Let $\chi \in \mathcal{D}(\Omega_2)$ be a non-negative test function with $\chi = 1$ in Ω_1. Then

$$\int_{\Omega_1} T \wedge dd^c u \le \int_{\Omega_2} \chi T \wedge dd^c u.$$

Since $dd^c\chi = 0$ in Ω_1 we obtain

$$\int_{\Omega_2} \chi dd^c u \wedge T = \int_{\Omega_2} u dd^c \chi \wedge T = \int_{\Omega_2\setminus\Omega_1} u dd^c \chi \wedge T.$$

Fixing $A > 0$ such that $dd^c\chi \ge -A\beta$ on Ω we infer

$$\int_{\Omega_2} \chi dd^c u \wedge T \le A\|u\|_{\Omega_2\setminus\Omega_1} \int_{\Omega_2\setminus\Omega_1} T \wedge \beta.$$

We now prove the second inequality (3.2). We can assume without loss of generality that $u := u_1 \ge 0$ in Ω_2. Then u^2 is plurisubharmonic and

$$dd^c u^2 = 2u dd^c u + 2du \wedge d^c u,$$

in the sense of currents in Ω_2. Thus (3.2) follows from (3.1) applied with u_1 replaced bu u_1^2,

$$\int_{\Omega_1} du_1 \wedge d^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k \wedge T \le C\|u_1\|_E^2 \ldots \|u_k\|_E \|T\|_E.$$

□

COROLLARY 3.14. *For all subdomains $\Omega_1 \Subset \Omega_2 \Subset \Omega$, there exists a constant $C = C_{\Omega_1,\Omega_2} > 0$ such that if $V \in PSH(\Omega)$, and $u_1,\ldots,u_k \in PSH(\Omega)\cap L^\infty_{loc}(\Omega)$, then*

$$\int_{\Omega_1} dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge dd^c V \wedge \beta_q \le C\|u_1\|_E \ldots \|u_k\|_E \|V\|_{L^1(E)}, \tag{3.3}$$

where $E := \Omega_2 \setminus \Omega_1$ *and* $q = n - (k+1)$.

PROOF. It suffices to find an upper bound for the mass of the current $T = dd^c V \wedge \beta_{n-1}$ on Ω_1 and apply previous inequalities.

We use the same notations as in the previous proof and assume first that $V < 0$ in Ω_2. Then

$$\int_{\Omega_1} dd^c V \wedge \beta_{n-1} \leq \int_{\Omega_2} \chi dd^c V \wedge \beta_{n-1} = \int_{\Omega_2} V dd^c \chi \wedge \beta_{n-1}.$$

Since $dd^c \chi \wedge \beta_{n-1} = \Delta \chi \beta_n$, it follows that

$$\int_K dd^c V \wedge \beta_{n-1} \leq \|\Delta \chi\|_{\Omega_2 \setminus \Omega_1} \int_{\Omega_2 \setminus \Omega_1} |V| \beta_n.$$

which proves the required inequality.

We now treat the the general case. The submean-value inequality insures that we can find an open set Ω' such that $\Omega_1 \Subset \Omega' \Subset \Omega_2$ and a constant $C > 0$ such that $\max_{\bar{\Omega}'} V \leq C \int_{\Omega_2 \setminus \Omega_1} V_+ \; \beta_n$. We can now apply the last inequality to $V - \sup_{\Omega'} V$ in Ω' and obtain the required estimate. □

3.2. Symmetry of Monge-Ampère operators.

PROPOSITION 3.15. *Let T be a closed positive current on Ω of bidimension (m, m). Let (u_j) be bounded plurisubharmonic functions in Ω which decrease to $u \in PSH(\Omega) \cap L^\infty_{loc}$. Then for any continuous plurisubharmonic function h on Ω,*

$$h dd^c u_j \wedge T \to h dd^c u \wedge T$$

and

$$dd^c h \wedge dd^c u_j \wedge T \to dd^c h \wedge dd^c u \wedge T$$

in the weak sense of Radon measures on Ω.

PROOF. We already know that $dd^c u_j \wedge T \to dd^c u \wedge T$ in the weak sense of currents by Proposition 3.7.

The Chern-Levine-Nirenberg inequalities insure that the currents $dd^c u_j \wedge T$ have locally uniformly bounded masses in Ω, hence the weak convergence holds in the sense of Radon measures on Ω. Thus

$$h dd^c u_j \wedge T \to h dd^c u \wedge T$$

in the weak sense of Radon measures.

Since the operator dd^c is continuous for the weak convergence of currents, it follows that $dd^c h \wedge dd^c u_j \wedge T \to dd^c h \wedge dd^c u \wedge T$ in the sense of currents in Ω.

Now these are positive currents (h is plurisubharmonic) hence the convergence actually holds in the sense of Radon measures. □

This shows that these operators are symmetric:

COROLLARY 3.16. *Let T (resp. S) be a positive closed current of bidimension $(2,2)$ (resp. (k,k)) and u, v be locally bounded plurisubharmonic functions in a domain $\Omega \subset \mathbb{C}^n$. Then*

$$dd^c u \wedge dd^c v \wedge T = dd^c v \wedge dd^c u \wedge T.$$

More generally the Monge-Ampère type operator

$$(u_1, \ldots, u_k) \in PSH(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) \longmapsto dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge S$$

is symmetric.

PROOF. The formula is clear when both functions are smooth. Assume now that u is smooth and take a sequence of smooth psh functions v_j which locally decrease to u. Then

$$dd^c u \wedge dd^c v_j \wedge T = dd^c v_j \wedge dd^c u \wedge T$$

hence the previous convergence result yields the required identity.

We now treat the general case. Since the property is local we can use the localization principle and assume that u, v are negative in a ball $\mathbb{B} \Subset \Omega$ and equal to 0 on $\partial\mathbb{B}$. Let ρ be a strictly plurisubharmonic defining function of $\mathbb{B}$. It follows from Proposition 3.11 that

$$\int_{\mathbb{B}} \rho dd^c u \wedge dd^c v \wedge T = \int_{\mathbb{B}} u dd^c \rho \wedge dd^c v \wedge T.$$

As ρ is smooth, the first part of the proof yields $dd^c \rho \wedge dd^c v \wedge T = dd^c v \wedge dd^c \rho \wedge T$ in the sense of currents. Thus

$$\int_{\mathbb{B}} \rho dd^c u \wedge dd^c v \wedge T = \int_{\mathbb{B}} u dd^c v \wedge dd^c \rho \wedge T.$$

Using again the formula (2.8) and the fact that $dd^c \rho \wedge dd^c u \wedge T = dd^c u \wedge dd^c \rho \wedge T$, we get

$$\begin{aligned} \int_{\mathbb{B}} \rho dd^c u \wedge dd^c v \wedge T &= \int_{\mathbb{B}} v dd^c u \wedge dd^c \rho \wedge T \\ &= \int_{\mathbb{B}} v dd^c \rho \wedge dd^c u \wedge T \\ &= \int_{\mathbb{B}} \rho dd^c v \wedge dd^c u \wedge T. \end{aligned}$$

Observe finally that any smooth test function χ with compact support in $\mathbb{B}$ can be written as the difference of two defining functions for $\mathbb{B}$. Indeed $A dd^c \rho > -dd^c \chi$ for $A > 1$ large enough, thus $\chi = \rho_1 - \rho_2$, where $\rho_1 := \chi + A\rho$ and $\rho_2 := A\rho$. □

Here is a first example of an explicit non-smooth complex Monge-Ampère measure:

Example 3.17. *Fix $r > 0$. The function*

$$u_r(z) := \log^+(|z|/r) = \max\{\log|z|; \log r\} - \log r$$

is a continuous plurisubharmonic function in $\mathbb{C}^n$ which is smooth in $\mathbb{C}^n \setminus \{|z| = r\}$ where it satisfies $(dd^c u)^n = 0$.

The Borel measure $(dd^c u_r)^n$ is invariant under unitary transformations and has total mass 1 in $\mathbb{C}^n$ (see Proposition 3.52). It coincides with the normalized Lebesgue measure σ_r on the sphere $|z| = r$.

3.3. Integrability with respect to Monge-Ampère measures.

Theorem 3.18. *Fix $V \in PSH(\Omega)$ and $u_1, \dots, u_n \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. For any subdomain $D \Subset \Omega$ and any compact subset $K \subset D$, there exists a constant $C = C(K, D) > 0$ such that*

$$\int_K |V| dd^c u_1 \wedge \ldots \wedge dd^c u_n \leq C \|u_1\|_D \ldots \|u_n\|_D \|V\|_{L^1(D)}.$$

In particular the Monge-Ampère measure $dd^c u_1 \wedge \ldots \wedge dd^c u_n$ does not charge the polar set $P = \{V = -\infty\}$.

Proof. Since K is compact, we can cover it by finitely many small balls. Using the localization principle we can thus assume that there are euclidean balls such that $K \Subset \mathbb{B}_1 \Subset \mathbb{B} \Subset \Omega$ and that u_k coincides in a neighborhood of $\mathbb{B} \setminus \mathbb{B}_1$ with $A_k \rho$. Here ρ denotes a defining function for the ball $\mathbb{B}$ and $A_k \leq C\|u_k\|_{L^\infty(\mathbb{B})}$ with a uniform constant C.

We first assume that $V < 0$ on $\mathbb{B}$ and set $V_j := \sup\{V, j\rho\}$, for $j \in \mathbb{N}$. The (V_j)'s are bounded plurisubharmonic functions on $\mathbb{B}$ with boundary values 0 which converge to V. It follows from Proposition 3.11 that

$$\begin{aligned}
\int_K (-V_j) \wedge_{1 \leq k \leq n} dd^c u_k &\leq \int_{\mathbb{B}} (-V_j) \wedge_{1 \leq k \leq n} dd^c u_k \\
&= \int_{\mathbb{B}} (-u_1) dd^c V_j \wedge dd^c u_2 \ldots \wedge dd^c u_n.
\end{aligned}$$

The Chern-Levine-Nirenberg inequalities (3.1) yield

$$\int_K (-u_1) dd^c V_j \wedge dd^c u_2 \ldots \wedge dd^c u_n \leq C_n \Pi_{1 \leq k \leq n} \|u_k\|_{\mathbb{B}} \int_{\mathbb{B}} |V_j| d\lambda,$$

where $C_n > 0$ is a uniform constant.

On the other hand the formula (2.8) yields, setting $A = A_1 \cdots A_n$,

$$\begin{aligned}
\int_{\mathbb{B} \setminus \mathbb{B}_1} (-u_1) dd^c V_j \wedge dd^c u_2 \ldots \wedge dd^c u_n &= A \int_{\mathbb{B} \setminus \mathbb{B}_1} (-\rho) dd^c V_j \wedge \beta_{n-1} \\
&= A \int_{\mathbb{B}} (-V_j) dd^c \rho \wedge \beta_{n-1} \\
&\leq A \int_{\mathbb{B}} (-V_j) \beta_n.
\end{aligned}$$

Altogether this yields

$$\int_K (-V_j) dd^c u_1 \wedge \ldots \wedge dd^c u_n \leq (C + A) \int_{\mathbb{B}} (-V_j) \beta_n.$$

Since (V_j) decreases to $V \in L^1(\mathbb{B})$, the monotone convergence theorem implies

$$\int_K (-V) dd^c u_1 \wedge \ldots \wedge dd^c u_n \leq (C + A) \int_{\mathbb{B}} (-V) \beta_n < +\infty.$$

We can finally replace V by $V' := V - \sup_{\mathbb{B}} V$, note that $dd^c V = dd^c V'$, and use the submean value inequality to obtain

$$\|V'\|_{L^1(\mathbb{B})} \leq C_0 \|V\|_{L^1(\mathbb{B}_2)}$$

where $\mathbb{B} \Subset \mathbb{B}_2 \Subset \Omega$. This proves the required estimate. □

3.4. Compact singularities. The estimates (3.1) and (3.2) only require a control on the functions near the boundary of Ω. We can thus improve the last estimate as follows:

PROPOSITION 3.19. *Let T be a closed positive current of bidimension (p, p) and $\varphi \in PSH(\Omega)$. Assume there exists a compact set E and a strictly pseudoconvex domain D such that $E \subset D \subset \Omega$ and a constant $M > 0$ such that $-M \leq \varphi \leq 0$ on $\Omega \setminus E$. Then there exists a constant $C > 0$ which does not depend on φ and M such that*

$$\int_D |\varphi| T \wedge \beta_p \leq CM \int_{D \setminus E} T \wedge \beta_p.$$

In particular

$$dd^c \varphi \wedge T := dd^c(\varphi T),$$

is a well defined closed positive current in Ω.

PROOF. Let ρ be a defining function for D which is stricly plurisubharmonic in a neighborhood of $\bar{D}$. We can assume that $\beta \leq dd^c \rho$ on D. Set $\varphi_j := \max\{\varphi, -j\}$ and observe that φ_j is plurisubharmonic, bounded on D and $\varphi_j = \varphi$ on $D \setminus E$ for $j > M$. It follows from Proposition 3.11 that

$$\begin{aligned} \int_D (-\varphi_j) T \wedge \beta_p &\leq \int_D (-\varphi_j) dd^c \rho \wedge T \wedge \beta_{p-1} \\ &= \int_D (-\rho) dd^c \varphi_j \wedge T \wedge \beta_{p-1} \\ &\leq \sup_D (-\rho) \int_D dd^c \varphi_j \wedge T \wedge \beta_{p-1}. \end{aligned}$$

Since $\varphi_j = \varphi$ on $\Omega \setminus D$ for $j \geq M$, Chern-Levine-Nirenberg inequalities show that the last integral is bounded from above by $C_1 \cdot M$. Letting $j \to \infty$ yields the required estimate. □

Corollary 3.20. *The Monge-Ampère measure*

$$dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_n$$

is well defined for all plurisubharmonic functions which are locally bounded near the boundary of Ω.

Such plurisubharmonic functions are called psh functions with *compact singularities.*

Example 3.21. *The function*

$$\ell(z) := \log |z|, z \in \mathbb{C}^n$$

is plurisubharmonic in $\mathbb{C}^n$ *and has an isolated singularity at the origin. Its Monge-Ampère measure* $(dd^c\ell)^n$ *is therefore well defined. The reader will check in Exercise 3.15 that*

$$(dd^c\ell)^n = \delta_0,$$

is the Dirac mass at the origin, hence ℓ *is a fundamental solution of the Monge-Ampère operator. This latter notion is however of limited interest as this operator is non-linear when* $n \geq 2$ *and there are very many fundamental solutions !*

4. Continuity of the complex Monge-Ampère operator

When the (complex) dimension is $n = 1$ the complex Monge-Ampère operator is nothing but the Laplace operator dd^c. It is then a linear operator which is well defined on all subharmonic functions and is continuous for the weak (L^1_{loc}) convergence.

The situation is much more delicate in dimension $n \geq 2$. The complex-Monge-Ampère operator is then non linear and can not be defined for all plurisubharmonic functions. It is moreover discontinous for the the L^1_{loc}-convergence. We will nevertheless show that it is continuous for the monotone convergence.

4.1. Continuity along decreasing sequences. We first show that the complex Monge-Ampère operators is continuous along decreasing sequences of plurisubharmonic functions.

Theorem 3.22. *Let* T *be a closed positive current of bidegree* (p, p) *and let* $(u_0^j)_{j\in\mathbb{N}}, ..., (u_q^j)_{j\in\mathbb{N}}$ *be decreasing sequences of plurisubharmonic functions which converge to* $u_0, ..., u_q \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$, $p + q \leq n$. *Then*

$$u_0^j \, dd^c u_1^j \wedge \ldots \wedge dd^c u_q^j \wedge T \longrightarrow u_0 \, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$$

in the weak sense of currents.

Proof. The proof proceeds by induction on q. For $q = 0$ the theorem is a consequence of the monotone convergence theorem. Fix $1 \leq q \leq n-p$ and assume that the theorem is true for $q-1$ so that

$$S^j := \wedge_{1 \leq k \leq q} dd^c u_k^j \wedge T \longrightarrow S := \wedge_{1 \leq k \leq q} dd^c u_k \wedge T.$$

It follows from Chern-Levine-Nirenberg inequalities (3.1) that the sequence $(u_0^j S^j)$ is relatively compact for the weak topology of currents. Up to extracting and relabelling, it suffices to show that if the sequence $(u_0^j S^j)$ converges weakly to a current Θ then $\Theta = u_0 S$.

By upper semi-continuity we already know that for all elementary positive $(n-p-q, n-p-q)$-form Γ, $\Theta \wedge \Gamma \leq u_0 S \wedge \Gamma$ hence $u_0 S - \Theta$ is a positive current on Ω. It thus remains to prove that

$$\int_\Omega u_0 S \wedge \beta^{n-p-q} \leq \int_\Omega \Theta \wedge \beta^{n-p-q}.$$

The problem being local, it is enough to prove that the total mass of the positive current $u_0 S - \Theta$ on each ball $\mathbb{B} = \mathbb{B}(a, R) \Subset \Omega$ is zero. By the localization principle we can assume that the functions u_0^j coincide with the function $\rho(z) = A(|z-a|^2 - R^2)$ in a neighborhood of $\partial\mathbb{B}$, where $A > 1$ is a large constant, and that $-1 \leq u_0^j < 0$ in an open neighborhood $\Omega'' \Subset \Omega$. Integrating by parts (using (2.8)), we infer

$$\begin{aligned}
\int_{\mathbb{B}} u_0 \wedge_{1 \leq i \leq q} dd^c u_i \wedge T \ \wedge \ \beta^{n-p-q} &\leq \int_{\mathbb{B}} u_0^j \wedge_{1 \leq i \leq q} dd^c u_i \wedge T \wedge \beta^{n-p-q} \\
&= \int_{\mathbb{B}} u_1 \wedge dd^c u_0^j \wedge_{2 \leq i \leq q} dd^c u_i \wedge T \wedge \beta^{n-p-q} \\
&\leq \int_{\mathbb{B}} u_0^j \wedge_{1 \leq i \leq q} dd^c u_i^j \wedge T \wedge \beta^{n-p-q}.
\end{aligned}$$

We use here the symmetry of the wedge products (see Corollary 3.16).

Since the positive measures $(-u_0^j) \wedge_{1 \leq i \leq q} dd^c u_i^j \wedge T \wedge \beta^{n-p-q}$ converge weakly to $-\Theta \wedge \beta^{n-p-q}$, it follows from the lower semi-continuity that

$$\liminf_{j \to +\infty} \int_{\mathbb{B}} (-u_0^j) \wedge_{1 \leq i \leq q} dd^c u_i^j \wedge \wedge T \wedge \beta^{n-p-q} \geq \int_{\mathbb{B}} -\Theta \wedge \beta^{n-p-q},$$

which proves the theorem. □

The reader can use decreasing sequence of smooth approximants to compute the following complex Monge-Ampère measure:

Example 3.23. *Consider*

$$\psi : z \in \mathbb{C}^n \mapsto \max\{\log^+ |z_j|; 1 \leq j \leq n\} \in \mathbb{R}.$$

The function ψ is Lipschitz continuous and plurisubharmonic in $\mathbb{C}^n$. Its Monge-Ampère measure $(dd^c\psi)^n$ coincides with the normalized Lebesgue measure τ_n on the torus

$$\mathbb{T}^n = \{z \in \mathbb{C}^n;\ |z_1| = \cdots = |z_n| = 1\}.$$

(See Exercise 3.9).

COROLLARY 3.24. *If u, v are plurisubharmonic and locally bounded, then*

$$(dd^c[u+v])^n = \sum_{j=0}^{n} \binom{n}{j} (dd^c u)^j \wedge (dd^c v)^{n-j}.$$

PROOF. The formula is clear if u, v are smooth. The general case follows by approximation by smooth decreasing sequences. □

The following estimates are due to Blocki [**Blo93**]:

PROPOSITION 3.25. *Fix $u, v, w \in PSH^-(\Omega) \cap L^\infty(\Omega)$ such that $\lim_{z \to \partial\Omega}(w(z) - v(z)) = 0$. Then*

$$\int_\Omega (w-v)_+^{n+1} (dd^c u)^n \leq (n+1)! M^{n+1} \int_\Omega (w-v)_+ \, (dd^c v)^n,$$

where $M = \sup_\Omega u - \inf_\Omega u$ and $(w-v)_+ := \sup\{w-v, 0\}$.

PROOF. We can assume without loss of generality that $\sup_\Omega u = 0$. Observe that for $\varepsilon > 0$ the function $w_\varepsilon := \sup\{v, w - \varepsilon\}$ is a bounded plurisubharmonic function on Ω such that $w_\varepsilon \nearrow \sup\{v, w\}$ as $\varepsilon \searrow 0$ and $w_\varepsilon = v$ near $\partial\Omega$. By the the monotone convergence theorem we may thus assume that $w = v$ near $\partial\Omega$.

Set $h := (w-v)_+$ on Ω and fix a compact set $K \subset \Omega$ such that $h = 0$ in $\Omega \setminus K$. Consider smooth approximants $h_\varepsilon := h \star \rho_\varepsilon$ of h. These functions are smooth in a neighborhood Ω'' of K with compact support in the $\varepsilon-$neighborhood K_ε of K. By definition $dd^c u \wedge T := dd^c(uT)$ hence

$$\int_\Omega h_\varepsilon^p dd^c u \wedge T = \int_{\Omega''} u dd^c h_\varepsilon^p \wedge T.$$

On the other hand if $p \geq 1$,

$$dd^c h_\varepsilon^p = p h_\varepsilon^{p-1} dd^c h_\varepsilon + p(p-1) h_\varepsilon^{p-2} dh_\varepsilon \wedge d^c h_\varepsilon \geq p h_\varepsilon^{p-1} dd^c h_\varepsilon,$$

thus $u \, dd^c h_\varepsilon^p \leq p u h_\varepsilon^{p-1} dd^c h_\varepsilon$ and

$$\begin{aligned} \int_{\Omega''} h_\varepsilon^p dd^c u \wedge T &\leq \int_{\Omega''} p u h_\varepsilon^{p-1} dd^c h_\varepsilon \wedge T \\ &\leq pM \int_{\Omega''} h_\varepsilon^{p-1} (-dd^c h_\varepsilon) \wedge T \\ &\leq pM \int_{\Omega''} h_\varepsilon^{p-1} dd^c v_\varepsilon \wedge T. \end{aligned}$$

The last inequality follows from the observation that

$$h = \max\{w-v, 0\} = \max\{w, v\} - v,$$

hence $-dd^c h_\varepsilon \leq dd^c v_\varepsilon$.

We can use this argument $n+1$ times and obtain

$$\int_{\Omega''} h_\varepsilon^{n+1}(dd^c u)^n \leq (n+1)! M^{n+1} \int_{\Omega''} h_\varepsilon (dd^c v_\varepsilon)^n.$$

Since $h = \sup\{w, v\} - v$, $h_\varepsilon = (\sup\{w, v\})_\varepsilon - v_\varepsilon$ is the difference of decreasing sequences of bounded plurisubharmonic functions, we can use the continuity of the Monge-Ampère operator along decreasing sequences and apply Lebesgue's convergence theorem to obtain

$$\int_{\Omega''} h^{n+1}(dd^c u)^n \leq (n+1)! M^{n+1} \int_{\Omega''} h (dd^c v)^n,$$

which is our claim. □

4.2. Continuity along increasing sequences. In this section we show that complex Monge-Ampère operators are continuous along increasing sequences.

To this end we need the following technical result which relies on the quasicontinuity of plurisubharmonic functions:

LEMMA 3.26. *Let $\mathcal{P}$ be a family of plurisubharmonic functions which are locally uniformly bounded. Let $\mathcal{T}$ denote the set of currents of the form $T := \bigwedge_{1\leq i\leq p} dd^c u_i$, where $u_1, \ldots, u_n \in \mathcal{P}$.*

If $(T_j)_{j\in\mathbb{N}}$ is a sequence of currents in $\mathcal{T}$ converging weakly to a current $T \in \mathcal{T}$, then for all locally bounded plurisubharmonic function φ, the currents φT_j weakly converge to φT.

The proof of the quasi-continuity will be given in the next section so we postpone the proof of this Lemma as well.

THEOREM 3.27. *Let $(u_0^j), \ldots, (u_q^j)$ be sequences of locally bounded plurisubharmonic functions which increase almost everywhere towards $u_0, \ldots, u_q \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. Then*

$$u_0^j dd^c u_1^j \wedge \ldots \wedge dd^c u_q^j \longrightarrow u_0 dd^c u_1 \wedge \ldots \wedge dd^c u_q$$

in the weak sense of currents.

PROOF. We proceed by induction on q. The case $q = 0$ follows from the monotone convergence theorem.

Suppose that the theorem is true for $q-1$. By continuity of dd^c we infer that the currents $S_j := dd^c u_1^j \wedge \ldots \wedge dd^c u_q^j$ converge to the current $S := dd^c u_1 \wedge \ldots \wedge dd^c u_q$.

The Chern-Levine-Nirenberg inequalities insure that the currents $(u_0^j S_j)$ form a relatively compact sequence. We need to show it has a unique limit point. Extracting and relabelling, we need to show that if $u_0^j S_j \to \Theta$ weakly then $\Theta = u_0 S$ on Ω.

The problem is local so it suffices to prove the convergence in a ball $\mathbb{B} = \mathbb{B}(a; r) \Subset \Omega$. We can modify the functions in a neighborhood of $\partial\mathbb{B}$ so that they all coincide with $\rho(z) = A(|z-a|^2 - R^2)$ near $\partial\mathbb{B}$, $A > 1$ a uniform constant.

Note that $u_0^j S_j \leq u_0 S_j$ since (u_0^j) is increasing. It follows therefore from the upper semi-continuity that $\Theta \leq u_0 S$. We now show that $\Theta = u_0 S$ by proving that

$$\int_{\overline{\mathbb{B}}} \Theta \wedge \beta_{n-q} \geq \int_{\overline{\mathbb{B}}} u_0 S \wedge \beta_{n-q}.$$

Indeed for $j \geq k \geq 0$, the integration by parts formula yields

$$\int_{\mathbb{B}} u_0^j S_j \wedge \beta_{n-q} \geq \int_{\mathbb{B}} u_0^k \wedge_{1 \leq i \leq q} dd^c u_i^j \wedge \beta_{n-q}.$$

The induction hypothesis and Lemma 3.26 yield

$$\begin{aligned} \liminf_j \int_{\mathbb{B}} u_0^j S_j \wedge \beta_{n-q} &\geq \int_{\mathbb{B}} u_0^k \wedge_{1 \leq i \leq q} dd^c u_i \wedge \beta_{n-q} \\ &= \int_{\mathbb{B}} u_1 dd^c u_0^k \wedge_{2 \leq i \leq q} dd^c u_i \wedge \beta_{n-q}. \end{aligned}$$

Applying Lemma 3.26 and Stokes' theorem again, we get

$$\begin{aligned} \lim_k \int_{\mathbb{B}} u_1 dd^c u_0^k & \wedge_{2 \leq i \leq q} dd^c u_i \wedge \beta_{n-q} \\ &= \int_{\mathbb{B}} u_1 dd^c u_0 \wedge_{2 \leq i \leq q} dd^c u_i \wedge \beta_{n-q} \\ &= \int_{\mathbb{B}} u_0 \wedge_{1 \leq i \leq q} dd^c u_i \wedge \beta_{n-q}, \end{aligned}$$

hence

$$\liminf_{j \to +\infty} \int_{\mathbb{B}} u_0^j S_j \wedge \beta_{n-q} \geq \int_{\mathbb{B}} u_0 \wedge_{1 \leq i \leq q} dd^c u_i \wedge \beta_{n-q},$$

and the proof is complete. □

The following corollary will allow us to show in the next chapter that "negligible sets are pluripolar", answering a celebrated question of P.Lelong.

Corollary 3.28. *Let (V_j) be plurisubharmonic functions increasing almost everywhere to $V \in PSH(\Omega)$. Then the exceptional set*

$$N := \{\sup_j V_j < V\}$$

has measure 0 with respect to all Monge-Ampère measures of the type $dd^c u_1 \wedge \ldots dd^c u_n$, where $u_1, \ldots, u_n \in PSH(\Omega) \cap L^\infty(\Omega)$.

We have already seen (see Theorem 2.22) that

$$\sup_j V_j(x) = \limsup V_j(x) = V(x)$$

for almost every x with respect to Lebesgue measure. This corollary gives much more precise information.

PROOF. Set $V_s := \sup\{V, s\}$ and $V_{j,s} = \sup\{V_j, s\}$ and observe that

$$N \subset \bigcup_{s\in\mathbb{Q}} N_s, \text{ where } N_s := \{\sup_j V_{j,s} < V_s\}.$$

By the subadditivity property of Borel measures, we can therefore assume that all the functions (V_j) are locally bounded.

Set $W := \sup_j V_j$. Then W is a locally bounded Borel function such that $W \leq V$ with equality almost everywhere. Again by subadditivity, it is enough to prove that

$$\int_{K\cap N} dd^c u_1 \wedge \ldots dd^c u_n = 0,$$

where $K \subset \Omega$ is a compact subset. Pick χ a non-negative cutoff function such that $\chi \equiv 1$ near K. The previous convergence theorem yields

$$\begin{aligned} \int_\Omega \chi W dd^c u_1 \wedge \ldots dd^c u_n &= \lim_j \int_\Omega \chi V_j dd^c u_1 \wedge \ldots dd^c u_n \\ &= \int_\Omega \chi V dd^c u_1 \wedge \ldots dd^c u_n, \end{aligned}$$

which proves the required result since $W \leq V$. □

4.3. Discontinuity of the Monge-Ampère operator. We have proved that the complex Monge-Ampère operator is well defined on the set $PSH(\Omega)\cap L^\infty_{loc}$ and is continuous under monotone convergence. This operator is however not continuous under the weaker L^1_{loc} convergence as was first emphasized by Cegrell [**Ceg84**]. Here is a simple example:

EXAMPLE 3.29. *The functions*

$$u_j(z_1, z_2) := \frac{1}{2j} \log\left[|z_1^j + z_2^j|^2 + 1\right]$$

are smooth and plurisubharmonic in $\mathbb{C}^2$. *They form a locally bounded sequence which converges in* $L^1_{loc}(\mathbb{C}^2)$ *towards*

$$u(z_1, z_2) = \log\max[1, |z_1|, |z_2|].$$

Observe that $(dd^c u_j)^2 = 0$ *while* $(dd^c u)^2$ *is the Lebesgue measure on the real torus* $\{|z_1| = |z_2| = 1\}$.

We leave the details as an Exercise 3.10. Another example is given in Exercise 3.7. This discontinuity is actually rather common as was observed by Lelong [**Lel83**] who showed the following:

PROPOSITION 3.30. *Every locally bounded plurisubharmonic function can be approximated in* L^1_{loc} *by locally bounded plurisubharmonic functions with vanishing Monge-Ampère measures.*

Such plurisubharmonic functions are natural generalizations of harmonic functions, they are called *maximal plurisubharmonic* functions.

5. Quasi-continuity

We explained how quasi-continuity of a plurisubharmonic functions $u \in PSH(\Omega)$ may help to understand the complex Monge-Ampère measure of a bounded plurisubharmonic function u by approximating it by a decreasing sequence of smooth plurisubharmonic functions converging to u.

We may ask what kind of convergence (or topology) on $PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ makes the complex Monge-Ampère operator continuous ? This will be explained in this section.

5.1. The Monge-Ampère capacity. As we saw in Chapter 3, the polar locus (i.e. the $-\infty$ locus) of a plurisubharmonic function is a null set of several Borel measures. These small sets cannot be characterized by a single measure, one has to introduce *capacities*, a non-linear generalization of the latter.

Capacities play an important role in Complex Analysis as they allow to characterize small sets. There are various capacities, depending on the problem of study. We introduce here a generalized capacity in the sense of Choquet, which is invariant under holomorphic isomorphisms and whose null sets are pluripolar sets (i.e. sets that are locally contained in the polar locus of a plurisubharmonic function).

We follow the seminal work of Bedford and Taylor [**BT82**] (with subsequent simplifications by Cegrell [**Ceg88**] and Demailly [**Dem91**]).

In classical potential theory, the capacity is defined as the maximal amount of charge supported on a compact set $K \subset \Omega \subset \mathbb{R}^m$ so that the difference of potentials in the condenser (K, Ω) is 1. This definition can be formalized by setting

$$\mathrm{Cap}(K) := \sup\{\mu(K); \mu \in \Gamma(K)\},$$

where $\Gamma(K)$ is the set of Borel measures that are supported on K and whose potential U_μ is bounded between 0 and 1 in Ω. Here U_μ denotes the Green potential of μ, i.e. the (generalized) solution of the Dirichlet problem $\Delta u = \mu$ in $\Omega \setminus K$ with boundary values $u = -1$ on ∂K and 0 on $\partial\Omega$.

There is no formula for U_μ in $\mathbb{C}^n$ when $n \geq 2$, as the corresponding complex Monge-Ampère operator is non linear. We will nevertheless mimic the above definition, using the family of bounded plurisubharmonic functions with zero boundary values on $\partial\Omega$ as potentials.

5.1.1. *Definition.* Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain. This means that Ω admits a continuous negative plurisubharmonic exhaution ρ.

The Monge-Ampère capacity is defined as follows:

Definition 3.31. *For any Borel subset $E \subset \Omega$ we set*

$$\mathrm{Cap}(E;\Omega) := \sup\left\{\int_E (dd^c u)^n ; u \in PSH(\Omega), -1 \le u \le 0\right\}.$$

We shall also use the notation $\mathrm{Cap}_\Omega(E)$ in the sequel.

It follows from Chern-Levine-Nirenberg inequalities that the capacity of relatively compact subsets $E \Subset \Omega$ is finite, $\mathrm{Cap}(E;\Omega) < +\infty$.

The real number $\mathrm{Cap}(E;\Omega)$ is the (inner) Monge-Ampère capacity of the condenser (E,Ω). Since any Borel measure is inner regular, it follows that the set function $\mathrm{Cap}(\cdot;\Omega)$ is inner regular. The outer Monge-Ampère capacity is then:

Definition 3.32. *For a any subset $E \subset \Omega$, we set*

$$\mathrm{Cap}^*(E,\Omega) := \inf\{\mathrm{Cap}(G,\Omega),\ G \text{ open},\ E \subset G \subset \Omega\}.$$

We say that E is capacitable if $\mathrm{Cap}^*(E,\Omega) = \mathrm{Cap}(E,\Omega) < +\infty$.

We show hereafter, using Choquet's Theorem, that all Borel sets are capacitable. We start by establishing some elementary properties of this capacity:

Proposition 3.33.
1) If $\Omega \subset \mathbb{B}(a,R) \Subset \mathbb{C}^n$, then for any Borel subset $E \subset \Omega$,

$$\lambda_{2n}(E) \le \left(\frac{\pi R^2}{2}\right)^n \mathrm{Cap}^*(E;\Omega).$$

2) If $E_1 \subset E_2 \subset \Omega_2 \subset \Omega_1$, then $\mathrm{Cap}^*(E_1;\Omega_1) \le \mathrm{Cap}^*(E_2;\Omega_2)$.

3) The set function $\mathrm{Cap}^*(\cdot;\Omega)$ is subadditive, i.e. if $(E_j)_{j\in\mathbb{N}}$ is any sequence of subsets of Ω, then

$$\mathrm{Cap}^*(E;\Omega) \le \sum_j \mathrm{Cap}^*(E_j;\Omega).$$

4) Let $\Omega' \Subset \Omega'' \subset \Omega \subset \mathbb{C}^n$ be open subsets. Then there exists a constant $A = A(\Omega,\Omega',\Omega'') > 0$ such that for all $E \subset \Omega'$,

$$\mathrm{Cap}^*(E;\Omega) \le \mathrm{Cap}(E;\Omega'') \le A\mathrm{Cap}^*(E;\Omega).$$

5) Let $f : \Omega_1 \subset \mathbb{C}^n \longrightarrow \Omega_2 \subset \mathbb{C}^n$ be a proper holomorphic map. Then for any Borel subset $E \subset \Omega_2$ we have

$$\mathrm{Cap}(E,\Omega_2) \le \mathrm{Cap}(f^{-1}(E),\Omega_1).$$

Proof. The function $\rho(z) := |z-a|^2/R^2 - 1$ is plurisubharmonic in Ω and $-1 \le \rho \le 0$ in $\Omega \subset \mathbb{B}(a,R)$. By definition we thus get

$$\int_E (dd^c\rho)^n \le \mathrm{Cap}(E;\Omega),$$

which proves the first property since $dd^c\rho = (2/\pi R^2)\beta$.

We let the reader check the properties 2,3 and prove property 4. Fix ρ a plurisubharmonic defining function for Ω and $c > 0$ such that

$\Omega'' \subset \Omega_c := \{\rho < -c\}$, hence $\mathrm{Cap}(\cdot, \Omega'') \geq \mathrm{Cap}(\cdot, \Omega_c)$. It is sufficient to prove the inequality for Ω_c. Choose $A > 1$ so that $\psi := A(\rho + c)$ satisfies $\psi \leq -1$ on Ω'. Fix $u \in PSH(\Omega_c)$ with $-1 \leq u \leq 0$ and define

$$\tilde{u}(z) := \begin{cases} \max(u(z), \psi(z)) & \text{if} \quad z \in \Omega_c \\ \psi(z) & \text{if} \quad z \in \Omega \setminus \Omega_c \end{cases}$$

so that $\tilde{u}$ is plurisubharmonic on Ω and satisfies $\tilde{u} = u$ on Ω'.

The function $v := (a+1)^{-1}(\tilde{u} - a)$, where $a := \sup_\Omega \tilde{u}$, is plurisubharmonic in Ω and $-1 \leq v \leq 0$. Thus $\int_E (dd^c v)^n \leq \mathrm{Cap}(E; \Omega)$. Since $v = (a+1)^{-1}(u - a)$ in Ω', we infer

$$\int_E (dd^c u)^n \leq (a+1)^n \mathrm{Cap}(E; \Omega),$$

hence $\mathrm{Cap}(E; \Omega_c) \leq (a+1)^n \mathrm{Cap}(E; \Omega)$.

We finally prove property 5. Fix $u \in PSH(\Omega_2) \cap L^\infty_{loc}(\Omega_2)$. It follows from Exercise 3.5 in Chapter 3 that $f_*(dd^c u \circ f)^n = (dd^c u)^n$ in the sense of Borel measures in Ω_1. Thus if $E \subset \Omega_2$ is a Borel subset,

$$\int_E (dd^c u)^n = \int_{f^{-1}(E)} (dd^c u \circ f)^n \leq \mathrm{Cap}(f^{-1}(E), \Omega_1).$$

Taking the supremum over all such u yields the required inequality for the inner capacities. □

5.1.2. *Polar sets are null sets.* We now show that the polar locus of a plurisubharmonic function has zero outer capacity.

PROPOSITION 3.34. *Let $\Omega' \Subset \Omega \subset \mathbb{C}^n$ be two open sets and $K \subset \Omega'$ a compact set. There exists $A = A(K, \Omega') > 0$ such that for all plurisubharmonic functions $V \in PSH(\Omega)$ and for all $s > 0$,*

$$\mathrm{Cap}^*(\{z \in K; V(z) < -s\}; \Omega) \leq \frac{A}{s} \|V\|_{L^1(\Omega')}. \tag{5.1}$$

In particular the polar locus $P := \{z \in \Omega; V(z) = -\infty\}$ satisfies

$$\mathrm{Cap}^*(P, \Omega) = 0.$$

PROOF. Let $u \in PSH(\Omega)$ be so that $-1 \leq u \leq 0$. It follows from the Chern-Levine-Nirenbeg inequalities that there exists $A = A(K, \Omega') > 0$ such that

$$\int_K |V| (dd^c u)^n \leq A \int_{\Omega'} |V| d\lambda_{2n}.$$

We infer

$$\int_{\{z \in K; V(z) \leq -s\}} (dd^c u)^n \leq \frac{1}{s} \int_K |V| (dd^c u)^n \leq \frac{A}{s} \int_{\Omega'} |V| d\lambda_{2n}.$$

The desired estimate for the inner capacity follows.

Since for any open set $D \Subset \Omega$, the sublevel sets $\{z \in D; V < -s\}$ are open sets, it follows that the same inequality holds for the outer capacity. The last statement follows from subadditivity of the outer capacity. □

We will later on show that conversely, if $\mathrm{Cap}^*(P,\Omega)=0$, then P is (locally) contained in the polar set of a plurisubharmonic function.

5.2. Quasi-continuity of plurisubharmonic functions.

THEOREM 3.35. *Let V be a plurisubharmonic function in Ω. For all $\varepsilon>0$, there exists an open set $G\subset\Omega$ such that $\mathrm{Cap}(G,\Omega)<\varepsilon$ and $V|(\Omega\setminus G)$ is continuous.*

PROOF. Let $D\Subset\Omega$ be an open subset. It follows from Proposition 3.34 that $\mathrm{Cap}(G_1,\Omega)<\varepsilon$ if $s>1$ is large enough, where

$$G_1:=\{z\in D; V(z)<-s\}.$$

Set $v=v_s:=\sup\{V,-s\}$. The function v is plurisubharmonic and bounded in a neighborhood of $\overline{D}$, and $v=V$ in $\Omega\setminus G_1$. Let (v_j) a decreasing sequence of smooth plurisubharmonic functions in a neighborhood $\Omega'\Subset\Omega$ of $\overline{D}$ which converges to v. We can assume that $v_j=v=A\rho$ in a neighbourhood of $\partial\Omega'$. It follows from Proposition 3.25 that for all $\delta>0$,

$$\mathrm{Cap}_\Omega\left(D\cap\{v_j-v>\delta\}\right)\le\frac{(n+1)!}{\delta^{n+1}}\int_{\Omega'}(v_j-v)(dd^cv)^n.$$

The monotone convergence theorem therefore yields

$$\lim_j\mathrm{Cap}_\Omega\left(D\cap\{v_j-v>\delta\}\right)=0.$$

Thus for all $k\in\mathbb{N}^*$ there exists $j_k\in\mathbb{N}$ large enough so that

$$\mathrm{Cap}_\Omega\left(D\cap\{v_{j_k}-v>1/k\}\right)\le\varepsilon 2^{-k}.$$

We can assume that the sequence (j_k) is increasing. Set

$$G_2:=\bigcup_{k\ge1}\{v_{j_k}-v>1/k\}\text{ and }G:=G_1\cup G_2.$$

The sequence (v_{j_k}) decreases uniformly to v on the compact set $\overline{D}\setminus G_2$ and $\mathrm{Cap}_\Omega(G_2)\le\varepsilon$, thus the open set G satisfies the required properties: it has small capacity and $V=v$ is continuous in $D\setminus G$.

To complete the proof of the theorem we take an exhaustive sequence (D_j) of relatively compact domains such that $\bigcup_j D_j=\Omega$ and apply the first part of the proof to find a sequence (G_j) of open subsets of Ω such that $\mathrm{Cap}_\Omega(G_j)\le\varepsilon2^{-j}$ and $V|(D_j\setminus G_j)$ is continuous. We finally set $G:=\bigcup_j G_j$ and use the subadditivity of the capacity to conclude that $\mathrm{Cap}_\Omega(G)\le\varepsilon$ and $V|(\Omega\setminus G)$ is continuous. □

The following lemma has been used in Chapter 3:

LEMMA 3.36. *Let $\mathcal{P}$ be a locally uniformly bounded family of plurisubharmonic functions in Ω. Let $\mathcal{T}$ be the set of currents of the form $T := \bigwedge_{1\leq i\leq p} dd^c u_i$, where $u_1, \ldots, u_p \in \mathcal{P}$.*

Assume that $(T_j)_{j\in\mathbb{N}}$ is a sequence of currents in $\mathcal{T}$ converging weakly to $T \in \mathcal{T}$. Then for any locally bounded quasi-continuous function f in Ω, $fT_j \longrightarrow fT$ in the weak sense of currents in Ω.

The notion of quasi-continuous function is defined as follows:

DEFINITION 3.37. *A Borel function $f : \Omega \longrightarrow \overline{\mathbb{R}}$ is quasi-continuous if for all $\varepsilon > 0$ and all compact subsets $K \Subset \Omega$ there exists an open subset $G \subset \Omega$ such that $Cap(G, \Omega) \leq \varepsilon$ and $f|(K \setminus G)$ is continuous.*

It follows from Theorem 3.35 that any plurisubharmonic function is quasi-continuous. So are the differences of plurisubharmonic functions.

PROOF. Let Θ be a positive continuous test form of bidegree $(n-p, n-p)$ in Ω, $K \subset \Omega$ its compact support and fix $\varepsilon > 0$ small.

It follows from the quasi-continuity of f that there is an open subset $G \subset \Omega$ with $\mathrm{Cap}(G, \Omega) \leq \varepsilon$ such that $f|(K \setminus G)$ is continuous. Let g be a continuous function in Ω with compact support such that $g = f$ on $K \setminus G$. Then

$$\int_\Omega f(T_j\wedge\Theta - T\wedge\Theta) = \int_\Omega g(T_j\wedge\Theta - T\wedge\Theta) + \int_G (f-g)(T_j\wedge\Theta - T\wedge\Theta).$$

Observe that $\lim_{j\to+\infty} \int_\Omega g(T_j \wedge \Theta - T \wedge \Theta) = 0$ since T_j weakly converges towards T. We claim that the second term is $O(\varepsilon)$. Indeed, since $|f - g|$ is bounded by a constant $M > 0$ on the support of Θ,

$$|\int_G (f-g)(T_j \wedge \Theta - T \wedge \Theta)| \leq M \int_G (T_j \wedge \Theta + T \wedge \Theta).$$

Observe that $T_j \wedge \Theta \leq C_1 T_j \wedge \beta^{n-p}$ for some $C_1 > 0$, where $\beta = dd^c|z|^2$. Moreover there exists a uniform constant $C_2 > 0$ such that for all Borel subsets $E \subset \Omega$,

$$\int_E \bigwedge_{1\leq i\leq p} (dd^c v_i) \wedge \beta^{n-p} \leq C_2 \mathrm{Cap}(E, \Omega)$$

as $v_1, \cdots, v_p \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ with $|v_i| \leq A$ (see Exercise **??**). Therefore

$$\int_G T_j \wedge \Theta + T \wedge \Theta \leq 2C_1C_2\varepsilon,$$

and the proof is complete. □

5.3. Convergence in capacity. We have shown that the complex Monge-Ampère operator is continuous along monotone sequences of plurisubharmonic functions. We introduce here, following [**X96**], a more general notion of convergence which contains the above continuity properties as a particular case.

DEFINITION 3.38. *A sequence of Borel functions $(f_j)_{j\geq 0}$ converges in capacity to a Borel function f in Ω if for all $\delta > 0$ and all compact subsets $K \subset \Omega$,*

$$\lim_{j\to+\infty} \mathrm{Cap}^*(K \cap \{|f_j - f| \geq \delta\}, \Omega)) = 0.$$

Convergence in capacity implies convergence in L^1_{loc} (but the converse is not true):

LEMMA 3.39. *Let $(f_j)_{j\geq 0}$ be sequence of (locally) uniformly bounded Borel functions which converges in capacity to a Borel function f in Ω. Then $f_j \to f$ in $L^1_{loc}(\Omega)$.*

PROOF. Fix $K \subset \Omega$ a compact set and $\delta > 0$. Since the sequence $(f_j)_{j\in\mathbb{N}}$ and f are uniformly bounded in K (by some $M > 0$), we get

$$\int_K |f_j - f| d\lambda \leq 2M\lambda(K \cap \{|f_j - f| > \delta\}) + \delta\lambda(K).$$

By Proposition 3.33, the Lebesgue measure λ is dominated by capacity, hence the first term in the right hand side converges to 0. The claim follows since $\delta > 0$ is arbitrarily small. □

We note that convergence of monotone sequences of plurisubharmonic functions implies convergence in capacity.

PROPOSITION 3.40. *Let $(V_j)_{j\in\mathbb{N}} \subset PSH(\Omega)$ be a monotone sequence of plurisubharmonic functions which converges almost everywhere to $V \in PSH(\Omega)$. Then (V_j) converges to V in capacity.*

We treat here two rather different settings at once. If (v_j) is non increasing, then $v_j(x)$ converges towards $v(x)$ at all points x (see Chapter 2). When (v_j) is non decreasing, then $w = \sup_j v_j$ is usually not u.s.c. hence equality $w(x) = v(x)$ does not hold everywhere.

PROOF. By subadditivity and monotonicity of the capacity, it is enough to prove that for any euclidean ball $\mathbb{B} \Subset \Omega$, any compact $K \subset \mathbb{B}$ and any $\delta > 0$ we have

$$\lim_{j\to+\infty} \mathrm{Cap}^*_{\mathbb{B}}(K \cap \{|V_j - V| \geq \delta\}) = 0.$$

We use here the fact $\mathrm{Cap}^*_{\Omega}(\cdot) \leq \mathrm{Cap}^*_{\mathbb{B}}(\cdot)$.

We first reduce to the case where the sequence (V_j) is locally uniformly bounded. Indeed fix $s \in \mathbb{N}$ and define

$$V_j^s := \sup\{V_j, -s\},\ V^s := \sup\{V, -s\}.$$

Then

$$\{|V_j - V| \geq \delta\} \subset \{|V_j^s - V^s| \geq \delta\} \cup \{V \leq -s\} \cup \{V_j \leq -s\}$$

Now by (5.1) we have for any $s \geq 1$ and $j \geq 1$

$$\mathrm{Cap}^*_{\mathbb{B}}(K \cap \{V_j < -s\} \leq \frac{A}{s} \|V_j\|_{L^1(\mathbb{B})},$$

where $A > 0$ is a constant which does not depend on j. Therefore

$$\mathrm{Cap}_{\mathbb{B}}{}^*(K \cap \{|V_j - V| \geq \delta\}) \leq \mathrm{Cap}^*_{\mathbb{B}}(K \cap \{|V_j^s - V^s| \geq \delta\}) + \frac{A'}{s},$$

hence it suffices to treat the case of sequences of plurisubharmonic functions that are uniformly bounded.

Assume thus that $-M \leq V_j, V \leq +M$ in $\mathbb{B}$, for some $M > 0$. Using the localization principle (see Chapter 3), we can assume all the V_j's are equal in a neighborhood of $\partial\mathbb{B}$.

Fix $\varepsilon > 0$. The quasi-continuity property of plurisubharmonic functions and the subadditivity of the capacity insure that we can find an open set $G \subset \Omega$ such that $Cap(G, \mathbb{B}) \leq \varepsilon$ and all V_j's and V are continuous in $\bar{\mathbb{B}} \setminus G$. Since the sequence (V_j) is monotone, it follows from Dini's lemma that the convergence is uniform on the compact set $\bar{\mathbb{B}} \setminus G$.

Assume first that $(V_j)_{j\in\mathbb{N}}$ is a non-decreasing sequence. It follows from Chebyshev inequality and Proposition 3.25 that

$$\begin{aligned}
&\frac{1}{(n+1)!}\mathrm{Cap}_{\mathbb{B}}(K \cap \{V - V_j \geq \delta\}) \\
\leq\ & \delta^{-n-1}\int_{\mathbb{B}}(V - V_j)(dd^cV_j)^n \\
\leq\ & \delta^{-n-1}\int_{\mathbb{B}\setminus G}(V - V_j)(dd^cV_j)^n + \delta^{-n-1}\int_{\mathbb{B}\cap G}(V - V_j)(dd^cV_j)^n \\
\leq\ & \delta^{-n-1}\|V_j - V\|_{\bar{\mathbb{B}}\setminus G}\int_{\mathbb{B}}(dd^cV_j)^n + 2M\delta^{-n-1}\int_G(dd^cV_j)^n,
\end{aligned}$$

Now $\int_{\mathbb{B}}(dd^cV_j)^n$ is uniformly bounded by Chern-Levine-Nirenberg inequalities and $\int_G(dd^cV_j)^n \leq M^n\mathrm{Cap}(G, \mathbb{B}) \leq \varepsilon M^n$. The conclusion follows since $\lim_j \|V_j - V\|_{\bar{\mathbb{B}}\setminus G} = 0$.

Assume now that $(V_j)_{j\in\mathbb{N}}$ is non-increasing. We proceed as above to obtain

$$\mathrm{Cap}_{\mathbb{B}}(K \cap \{V_j - V| \geq \delta\}, \mathbb{B}) \leq \delta^{-n-1}\,(n+1)!\int_{\mathbb{B}}(V_j - V)(dd^cV)^n.$$

The conclusion follows from the monotone convergence theorem and the capacitability of Borel sets (Corollary **??**). □

5.4. Continuity of the Monge-Ampère operator. Our aim in this section is to show that the complex Monge-Ampère operator is continuous for the convergence in capacity.

THEOREM 3.41. *Let $(f_j)_{j\in\mathbb{N}}$ be positive and uniformly bounded quasi-continuous functions which converge in capacity to a quasi-continuous function f in Ω. Let $(u_j^1)_{j\in\mathbb{N}}$, ..., $(u_j^p)_{j\in\mathbb{N}}$ be uniformly bounded plurisubharmonic functions which converge in capacity in Ω to locally bounded* *plurisubharmonic functions $u^1, ..., u^p$ respectively. Then*

$$f_j \bigwedge_{1\leq j\leq p} dd^c u_j^i \longrightarrow f \bigwedge_{1\leq j\leq p} dd^c u^i$$

in the weak sense of currents in Ω.

PROOF. The proof proceeds by induction. It follows from Lemma 3.39 that f_j (resp. u_j) converges in $L^1_{loc}(\Omega)$ towards f (resp. u). If $p = 1$, we know that $dd^c u_j^1 \to dd^c u^1$ weakly. Using induction on p and setting

$$T := \bigwedge_{1\leq i\leq p} dd^c u^i, \;\; T_j := \bigwedge_{1\leq i\leq p} dd^c u_j^i,$$

it is enough to prove that if $T_j \to T$ weakly then $f_j T_j \to fT$ weakly. The statement is local so we can assume that Ω is the unit ball and all the functions are bounded between -1 and 0. Fix Θ a test form of bidegree $(n-p, n-p)$ and observe that

$$\int_\Omega f_j T_j \wedge \Theta - \int_\Omega fT \wedge \Theta = \int_\Omega (f_j - f) T_j + \int_\Omega f\,(T_j \wedge \Theta - T \wedge \Theta) = I_j + J_j.$$

It follows from Lemma 3.36 that $\lim_j J_j = 0$. It thus remains to prove that $\lim_j I_j = 0$. Fix $\delta > 0$ small, let K be the support of Θ and set $E_j := K \cap \{|f_j - f| \geq \delta\}$. Then

$$|I_j| \leq \int_\Omega |f_j - f| T_j \wedge \Theta \leq \int_{K\cap E_j} |f_j - f| T_j \wedge \Theta + \delta \int_K T_j \wedge \Theta.$$

It follows from previous estimates that

$$|I_j| \leq C' \mathrm{Cap}^*(E_j, \Omega) + \delta M(K),$$

where $M(K) := C \sup_j \int_K T_j \wedge \beta^{n-p}$ is finite by Chern-Levine-Nirenberg inequalities, thus $\lim_j I_j = 0$. □

COROLLARY 3.42. *Let $(h_j)_{j\geq 0}$ and $(u_j)_{j\geq 0}$ be monotone sequences of uniformly bounded negative plurisubharmonic functions which converge almost everywhere to plurisubharmonic functions h and u respectively. Then for all $p \geq 0$,*

$$(-h_j)^p (dd^c u_j)^n \longrightarrow (-h)^p (dd^c u)^n.$$

PROOF. It follows from Proposition 3.40 that the sequences (u_j) and (h_j) converge in capacity to u and h respectively.

Observe that if $u \in PSH^-(\Omega)$ then $(-u)^p$ is quasicontinuous on Ω. Thus the sequence $(-h_j)^p$ converges in capacity to $(-h)^p$. The conclusion follows therefore from the previous theorem. □

COROLLARY 3.43. *Let $\mathcal{P}$ be a family of plurisubharmonic functions in a domain $\Omega \subset \mathbb{C}^n$ which is locally uniformly bounded from above and set $U := \sup\{u; u \in \mathcal{P}\}$. Then the exceptional set*

$$E := \{U < U^*\}$$

is negligible with respect to any measure $dd^c u_1 \wedge \cdots \wedge dd^c u_n$*, where* $u_1, \cdots, u_n \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$*. In particular*

$$\mathrm{Cap}(E, \Omega) = 0.$$

Proof. By Choquet's lemma (see Lemma 2.16 below) we reduce to the case when $\mathcal{P}$ is an increasing sequence of plurisubharmonic functions. Fix (v_j) be a sequence of locally uniformly bounded plurisubharmonic functions which increases almost everywhere to v and set

$$E := \{\sup_j v_j < v\}.$$

It follows from Corollary 3.42 that

$$v_j dd^c u_1 \wedge \cdots \wedge dd^c u_n \to v dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

weakly in Ω. Thus the positive currents $(v - v_j) dd^c u_1 \wedge \cdots \wedge dd^c u_n$ converge to 0, hence for any compact subset $K \subset \Omega$,

$$\lim_j \int_K (v - v_j) dd^c u_1 \wedge \cdots \wedge dd^c u_n = 0.$$

On the other hand if we set $w := \lim_j v_j$ in Ω, then $w \leq v$ and the monotone convergence theorem yields

$$\lim_j \int_K (w - v_j) dd^c u_1 \wedge \cdots \wedge dd^c u_n = 0.$$

Therefore $\int_K (w - v) dd^c u_1 \wedge \cdots \wedge dd^c u_n = 0$, hence $w = v$ almost everywhere in K for the measure $dd^c u_1 \wedge \cdots \wedge dd^c u_n$. □

Theorem 3.44. *Let* $(u_1^j)_{j\geq 0}, \ldots, (u_q^j)_{j\geq 0}$ *be decreasing sequences of bounded plurisubharmonic functions in* Ω *which converge respectively to* $u_1, \ldots, u_q \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$*. Let* (V_j) *be a decreasing sequence of plurisubharmonic functions which converges to* $V \in PSH(\Omega)$*. Then*

$$V_j dd^c u_1^j \wedge \ldots \wedge dd^c u_q^j \longrightarrow V dd^c u_1 \wedge \ldots \wedge dd^c u_q.$$

Proof. We already know that $dd^c u_1^j \wedge \ldots \wedge dd^c u_q^j \longrightarrow dd^c u_1 \wedge \ldots \wedge dd^c u_q$. It follows from Chern-Levine-Nirenberg inequalities that the currents $V_j dd^c u_1^j \wedge \ldots \wedge dd^c u_q^j$ have locally uniformly bounded masses.

Assume that $V_j \wedge_{1\leq i\leq q} dd^c u_i{}^j \wedge \beta^{n-p-q} \to \Theta$. The reader can check that $\Theta \leq V dd^c u_1 \wedge \ldots \wedge dd^c u_q$ (using an argument that we have already used in previous such proofs) and the problem is to show that $\int_{\mathbb{B}} V \wedge_{1\leq i\leq q} dd^c u_i \wedge \beta^{n-p-q} \leq \int_{\mathbb{B}} \Theta$, for an arbitrary ball $\mathbb{B}$.

We can assume all the $V_j's$ are negative and set $V_j^k := \sup\{V_j, k\rho\}$ for $k \in \mathbb{N}$, where ρ is a defining function for the ball $\mathbb{B}$. Since $V \leq$

$V_j \leq V_j^k$ on $\mathbb{B}$ and $V_j^k = 0$ in $\partial\mathbb{B}$, we get

$$\begin{aligned}\int_{\overline{\mathbb{B}}} V \wedge_{1\leq i\leq q} dd^c u_i \wedge \beta^{n-q} &\leq \int_{\overline{\mathbb{B}}} V_j^k \wedge_{1\leq i\leq q} dd^c u_i \wedge \beta^{n-q} \\ &= \int_{\overline{\mathbb{B}}} u_1 \wedge dd^c V_j^k \wedge_{2\leq i\leq q} dd^c u_i \wedge \beta^{n-q} \\ &\leq \int_{\overline{\mathbb{B}}} {u_1}^j \wedge dd^c V_j^k \wedge_{2\leq i\leq q} dd^c u_i \wedge \beta^{n-q} \\ &\leq \ldots \leq \int_{\overline{\mathbb{B}}} V_j^k \wedge_{1\leq i\leq q} dd^c u_i^j \wedge \beta^{n-q},\end{aligned}$$

for all $j, k \in \mathbb{N}$. The monotone convergence theorem yields

$$\int_{\overline{\mathbb{B}}} V \wedge_{1\leq i\leq q} dd^c u_i \wedge \beta^{n-q} \leq \int_{\overline{\mathbb{B}}} V_j \wedge_{1\leq i\leq q} dd^c {u_i}^j \wedge \beta^{n-q},$$

for all $j \in \mathbb{N}$. Letting now $j \to +\infty$, we obtain

$$\int_{\overline{\mathbb{B}}} V \wedge_{1\leq i\leq q} dd^c u_i \wedge \beta^{n-q} \leq \limsup_{j\to+\infty} \int_{\overline{\mathbb{B}}} V_j \wedge_{1\leq i\leq q} dd^c {u_i}^j \wedge \beta^{n-q} \leq \int_{\overline{\mathbb{B}}} \Theta,$$

and the proof is complete. □

6. Maximum principles

The comparison principle is one the most effective tools in pluripotential theory. It is a non linear version of the classical maximum principle.

6.1. The comparison principle. We establish in this section several types of maximum principles starting with the following *local maximum principle*:

THEOREM 3.45. *Set* $T = dd^c w_1 \wedge \cdots \wedge dd^c w_{n-p}$, *where* $0 \leq p \leq n-1$ *and* $w_i \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$. *Then for all* $u, v \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$,

$$\mathbf{1}_{\{u>v\}} (dd^c \max\{u, v\})^p \wedge T = \mathbf{1}_{\{u>v\}} (dd^c u)^p \wedge T,$$

in the sense of Borel measures in Ω.

PROOF. Set $D := \{u > v\}$. If u is continuous then D is an open subset of Ω and $\max\{u, v\} = u$ in D hence

$$(dd^c \max\{u, v\})^p \wedge T = (dd^c u)^p \wedge T,$$

so our claim is easy in this case.

We now treat the general case. Let (u_j) a sequence of continuous plurisubharmonic functions decreasing to u. Since the problem is local we can assume that Ω is a ball and all functions are bounded and plurisubharmonic on a fixed neighborhood of $\overline{\Omega}$. We know

$$\mathbf{1}_{\{u_j>v\}} (dd^c \max\{u_j, v\})^p \wedge T = \mathbf{1}_{\{u_j>v\}} (dd^c u_j)^p \wedge T,$$

Set $f_j := (u_j - v)^+$ and $f = (u-v)^+$. The previous identity yields

$$f_j (dd^c \max\{u_j, v\})^p \wedge T = f_j (dd^c u_j)^p \wedge T,$$

in the weak sense of measures in Ω.

Observe that $f_j = \max\{u_j, v\} - v$, $f = \max\{u, v\} - v$ and the sequence $(\max\{u_j, v\})$ decreases to $\max\{u, v\}$. It follows therefore from Theorem 3.22 that

$$f (dd^c \max\{u, v\})^p \wedge T = f (dd^c u)^p \wedge T$$

in the sense of Borel measures on Ω.

Fix $\varepsilon > 0$. Since $1/(f+\varepsilon)$ is a bounded Borel function, we infer

$$\frac{f}{f+\varepsilon} (dd^c \max\{u, v\})^p \wedge T = \frac{f}{f+\varepsilon} (dd^c u)^p \wedge T.$$

Let $\varepsilon \searrow 0$ and observe that $f/(f+\varepsilon) \nearrow \mathbf{1}_{\{u>v\}}$ to conclude. □

Corollary 3.46. *With the same hypotheses as in the theorem,*

$$(dd^c \max\{u, v\})^p \wedge T \geq \mathbf{1}_{\{u \geq v\}} (dd^c u)^p \wedge + \mathbf{1}_{\{u<v\}} (dd^c v)^p \wedge T$$

in the sense of Borel measures in Ω.

The following is often called the comparison principle:

Theorem 3.47. *Assume* $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$ *are such that* $\liminf_{z \to \partial\Omega} (u(z) - v(z)) \geq 0$. *Then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

Proof. By assumption we can find a compact subset $K \subset \Omega$ and an arbitrarily small $\varepsilon > 0$ such that $\sup\{u, v - \varepsilon\} = u$ in $\Omega \setminus K$. Fix a domain Ω' such that $K \subset \Omega' \Subset \Omega$. Then

$$\int_{\Omega'} (dd^c \sup\{u, v-\varepsilon\})^n = \int_{\Omega'} (dd^c u)^n. \tag{6.1}$$

Indeed set $w := \sup\{u, v - \varepsilon\}$ and observe that $(dd^c w)^n - (dd^c u)^n = dd^c S$, where $S := w(dd^c w)^{n-1} - u(dd^c u)^{n-1}$ is a current of bidimension $(1,1)$. Since $w = u$ in $\Omega \setminus K$, we get $S = 0$ there, i.e. the support of the current S is contained in K. Pick a smooth test function χ on Ω' such that $\chi \equiv 1$ in a neighborhood of K, we conclude that

$$\int_{\Omega'} dd^c S = \int_{\Omega'} \chi dd^c S = \int_{\Omega'} S \wedge dd^c \chi = 0,$$

since $dd^c \chi = 0$ on the support of the current S. This proves (6.1).

We now apply Theorem 3.45 and (6.1) in Ω' to get

$$\begin{aligned}
\int_{\{u<v-\varepsilon\}} (dd^c v)^n &= \int_{\{u<v-\varepsilon\}} (dd^c \sup\{u, v-\varepsilon\})^n \\
&= \int_{\Omega'} (dd^c \sup\{u, v-\varepsilon\})^n - \int_{\{u\geq v-\varepsilon\}} dd^c \sup\{u, v-\varepsilon\})^n \\
&\leq \int_{\Omega'} (dd^c u)^n - \int_{\{u>v-\varepsilon\}} dd^c \sup\{u, v-\varepsilon\})^n \\
&= \int_{\Omega'} (dd^c u)^n - \int_{\{u>v-\varepsilon\}} (dd^c u)^n \\
&= \int_{\{u\leq v-\varepsilon\}} (dd^c u)^n \leq \int_{\{u<v\}} (dd^c u)^n.
\end{aligned}$$

The conclusion follows by letting $\varepsilon \searrow 0$. □

We now deduce the following *global maximum principle*:

Corollary 3.48. *Assume $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$ are such that $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \geq 0$. If $(dd^c u)^n \leq (dd^c v)^n$ then $v \leq u$ in Ω.*

Proof. For $\varepsilon > 0$ we set $v_\varepsilon := v + \varepsilon\rho$, where $\rho(z) := |z|^2 - R^2$ is choosen so that $\rho < 0$ on Ω. Then $\{u < v_\varepsilon\} \subset \{u < v\} \Subset \Omega$. The comparison principle yields

$$\int_{\{u<v_\varepsilon\}} (dd^c v_\varepsilon)^n \leq \int_{\{u<v_\varepsilon\}} (dd^c u)^n.$$

It follows from Corollary 3.24 that

$$(dd^c v_\varepsilon)^n \geq (dd^c v)^n + \varepsilon^n (dd^c \rho)^n \geq (dd^c u)^n + \varepsilon^n (dd^c \rho)^n$$

hence $\int_{\{u<v_\varepsilon\}} (dd^c \rho)^n = 0$. We infer that the set $\{u < v_\varepsilon\}$ has Lebesgue measure 0. Since $\{u < v\} = \bigcup_{j\geq 1}\{u < v_{1/j}\}$, it follows that the set $\{u < v\}$ has Lebesgue measure 0 as well, hence $v \leq u$ on Ω by the sub-mean value inequalities. □

We now prove the domination principle.

Corollary 3.49. *Fix $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $v \leq u$ on $\partial\Omega$. Assume that $v \leq u$ a.e. in Ω with respect to the measure $(dd^c u)^n$. Then $v \leq u$ in Ω.*

Proof. For $\varepsilon > 0$ we set $v_\varepsilon := v + \varepsilon\rho$, where $\rho(z) := |z|^2 - R^2$ is choosen so that $\rho < 0$ on Ω. Then $\{u < v_\varepsilon\} \subset \{u < v\} \Subset \Omega$. The comparison principle yields

$$\int_{\{u<v_\varepsilon\}} (dd^c v_\varepsilon)^n \leq \int_{\{u<v_\varepsilon\}} (dd^c u)^n \leq \int_{\{u<v\}} (dd^c u)^n = 0.$$

Since $(dd^c v_\varepsilon)^n \geq \varepsilon^n (dd^c \rho)^n$, it follows that the set $\{u < v_\varepsilon\}$ has volume zero in Ω hence it is empty by the submean-value inequality. Therefore $\{u < v\}$ is empty, i.e. $v \leq u$ in Ω. □

6.2. The Lelong class. We have defined above the complex Monge-Ampère measure of plurisubharmonic functions that are bounded or with compact singularities.

One can not expect to define the Monge-Ampère measure of *any* unbounded plurisubharmonic function as the following example due to Kiselman [**Kis83**] shows:

EXAMPLE 3.50. *Set*

$$\varphi(z) := (-\log|z_1|)^{1/n}\left(|z_2|^2 + \cdots + |z_n|^2 - 1\right).$$

We let the reader check in Exercise 3.8 that φ *is a smooth plurisubharmonic function in* $\mathbb{B}^n \setminus \{z_1 = 0\}$ *such that*

$$(dd^c\varphi)^n = c_n \frac{1 - \frac{1}{n} - \sum_{\ell=2}^n |z_\ell|^2}{|z_1|^2 |\log|z_1||} dV_{Leb}$$

and that this measure has infinite mass in $\mathbb{B}^n \setminus \{z_1 = 0\}$.

We now introduce an important class of plurisubharmonic functions in $\mathbb{C}^n$ for which such a phenomenon cannot occur.

DEFINITION 3.51. *The Lelong class* $\mathcal{L}(\mathbb{C}^n)$ *is the class of plurisubharmonic functions* u *in* $\mathbb{C}^n$ *with logarithmic growth, i.e. for which there exists* $C_u \in \mathbb{R}$ *such that for all* $z \in \mathbb{C}^n$,

$$u(z) \leq \log^+|z| + C_u.$$

The reader will check in Exercise 3.13 that a non constant plurisubharmonic function in $\mathbb{C}^n$ has at least logarithmic growth. This class of functions will play an important role later as it induces the model class of quasi-plurisubharmonic functions on the complex projective space $\mathbb{CP}^n$.

We also consider

$$\mathcal{L}^+(\mathbb{C}^n) := \{u \in \mathcal{L}(\mathbb{C}^n) \,|\, \exists C'_u \text{ s.t. } -C'_u + \log^+|z| \leq u(z),\ \forall z \in \mathbb{C}^n\}.$$

We let the reader check in Exercise 3.14 that locally bounded functions from the Lelong class have finite total Monge-Ampère mass:

PROPOSITION 3.52. *If* u *belongs to* $\mathcal{L}(\mathbb{C}^n) \cap L^\infty_{loc}(\mathbb{C}^n)$ *then*

$$\int_{\mathbb{C}^n} (dd^c u)^n \leq 1.$$

Moreover if u *belongs to* $\mathcal{L}^+(\mathbb{C}^n)$, *then*

$$\int_{\mathbb{C}^n} (dd^c u)^n = 1.$$

The proof of these facts relies on the following result of independent interest:

LEMMA 3.53. *Let u, v be locally bounded plurisubharmonic functions in $\mathbb{C}^n$ such that $u(z) \to +\infty$ as $z \to \infty$. Assume that $v(z) \leq u(z) + o(u(z))$ as $z \to +\infty$. Then*

$$\int_{\mathbb{C}^n} (dd^c v)^n \leq \int_{\mathbb{C}^n} (dd^c u)^n.$$

PROOF. Fix $\varepsilon > 0$. Our assumption insures that for $R > 1$ large enough, $v(z) \leq (1+\varepsilon)u(z)$ if $|z| \geq R$. The comparison principle yields

$$\begin{aligned}\int_{B_R \cap \{(1+\varepsilon)u < v\}} (dd^c v)^n &\leq (1+\varepsilon)^n \int_{B_R \cap \{(1+\varepsilon)u < v\}} (dd^c u)^n \\ &\leq (1+\varepsilon)^n \int_{\mathbb{C}^n} (dd^c u)^n.\end{aligned}$$

Letting $R \to +\infty$ and $\varepsilon \to 0$ we obtain the required inequality. □

For classes of plurisubharmonic functions with prescribed Monge-Ampère mass, one might hope that it is easier to define the complex Monge-Ampère measure. This is however a delicate problem. The local domain of definition of the complex Monge-Ampère operator has been characterized by Blocki and Cegrell in [**Ceg04, Blo04, Blo06**].

7. Exercises

EXERCISE 3.1. *Let $(f_j)_{j \in \mathbb{N}}$ be a decreasing sequence of upper semi-continuous functions in a domain Ω converging to f. Let $(\mu_j)_{j \in \mathbb{N}}$ be a sequence of positive Borel measures in Ω which converges weakly to a Borel measure μ. Show that any limit point ν of the sequence of measures $\nu_j := f_j \cdot \mu_j$ satisfies the inequality $\nu \leq f \cdot \mu$ in the weak sense of Radon measures on Ω, i.e.*

$$\limsup_j f_j \mu_j \leq f\mu.$$

EXERCISE 3.2. *Let $(\mu_j)_{j \in \mathbb{N}}$ be a sequence of positive Borel measures on Ω which converges weakly to a positive Borel measure μ on Ω. Show that for any compact set $K \subset \Omega$ and any open subset $D \subset \Omega$,*

$$\limsup_j \mu_j(K) \leq \mu(K) \quad \textit{and} \quad \liminf_j \mu_j(D) \geq \mu(D).$$

EXERCISE 3.3. *Fix $0 < \alpha < 1$ and consider*

$$z \in \mathbb{D} \mapsto u(z) := -(1 - |z|^2)^\alpha \in \mathbb{R}.$$

Show that u is a smooth subharmonic function in the unit disc $\mathbb{D}$, which is Hölder continuous up to the boundary. Check that

$$\frac{\partial^2 u}{\partial z \partial \bar{z}}(z) = \alpha(1 - |z|^2)^{\alpha-1} + \alpha(1-\alpha)|z|^2(1 - |z|^2)^{\alpha-2},$$

and conclude that $\int_{\mathbb{D}} dd^c u = +\infty$. Is this in contradiction with Chern-Levine-Nirenberg inequalities ?

EXERCISE 3.4. *Let u be a locally bounded plurisubharmonic function in a domain Ω and $\chi : I \longrightarrow \mathbb{R}$ a smooth convex non-decreasing function with $u(\Omega) \subset I$. Check that $\varphi := \chi \circ u$ is plurisubharmonic in Ω with*

$$dd^c\varphi = \chi'(u)\, dd^c u + \chi''(u)\, du \wedge d^c u,$$

and

$$(dd^c\varphi)^n = \chi'(u)^n\, (dd^c u)^n + \chi''(u) \cdot \chi'(u)^{n-1}\, du \wedge d^c u \wedge (dd^c u)^{n-1}.$$

2. Fix $0 < \alpha < 1$ and consider

$$z \in \mathbb{B} \mapsto u(z) := -(1 - |z|^2)^\alpha \in \mathbb{R}.$$

Prove that u is is plurisubharmonic in the unit ball $\mathbb{B}$, Hölder continuous up to the boundary and satisfies $\int_{\mathbb{B}} (dd^c u)^n = +\infty$.

EXERCISE 3.5. *Let Ω, Ω' be domains in $\mathbb{C}^n$, $F : \Omega' \longrightarrow \Omega$ a holomorphic map and $u \in PSH(\Omega) \cap L^\infty(\Omega)$.*

1) If $u \in C^2(\Omega)$, show that

$$(dd^c u \circ F)^n(\zeta) = |J_F(\zeta)|^2 (dd^c u)^n(F(\zeta)),$$

as differential forms in Ω', where J_F denotes the Jacobian of F.

2) Check that if F is proper, then

$$F_*(dd^c u \circ F)^n = (dd^c u)^n$$

in the sense of currents.

3) Deduce that if F is a biholomorphism, then

$$(F^{-1})_*(dd^c u)^n = (dd^c u \circ F)^n.$$

EXERCISE 3.6. *Consider, for $1 \leq j \leq n$.*

$$z \in \mathbb{C}^n \mapsto u_j(z) := (Im z_j)^+ \in \mathbb{R}.$$

1) Check that these are continuous plurisubharmonic functions s.t.

$$dd^c u_1 \wedge \cdots \wedge dd^c u_n = (4\pi)^{-n} \iota_* \lambda_n,$$

in the sense of Borel measures on $\mathbb{C}^n$, where λ_n is the Lebesgue measure on $\mathbb{R}^n$ and $\iota : \mathbb{R}^n \longrightarrow \mathbb{C}^n$ is the embedding induced by $\mathbb{R} \to \mathbb{C}$.

2) Using Theorem 3.18 deduce that the restrictions of plurisubharmonic functions to $\mathbb{R}^n$ are locally integrable with respect to the n-dimensional Lebesgue measure λ_n on $\Omega \cap \mathbb{R}^n$.

3) Consider similarly $v : z \in \mathbb{C}^n \mapsto \sum_{j=1}^n (Im z_j)^+ \in \mathbb{R}^n$. Show that v is a Lipschitz continuous psh function in $\mathbb{C}^n$ such that

$$(dd^c v)^n = (n!/2\pi)^n \iota_* \lambda_n.$$

EXERCISE 3.7. *For $n \geq 2$ we set*

$$u_j(z) = \log(|z_1 \cdots z_n|^2 + 1/j) \text{ and } v_j(z) = \sum_{\ell=1}^n \log(|z_\ell|^2 + 1/j).$$

Show that these sequences of bounded plurisubharmonic functions both decrease to $\varphi(z) = 2\log|z_1 \cdots z_n|$ *and that* $(dd^c u_j)^n = 0$ *while* $(dd^c v_j)^n$ *converge to a positive multiple of the Dirac mass at the origin.*

EXERCISE 3.8. *Set*

$$\varphi(z) := (-\log|z_1|)^{1/n} \left(|z_2|^2 + \cdots + |z_n|^2 - 1\right).$$

1) Prove that φ *is a smooth psh function in* $\mathbb{B}^n \setminus \{z_1 = 0\}$ *such that*

$$(dd^c\varphi)^n = c_n \frac{1 - \frac{1}{n} - \sum_{\ell=2}^n |z_\ell|^2}{|z_1|^2 |\log|z_1||} dV_{Leb}$$

and that this measure has infinite mass in $\mathbb{B}^n \setminus \{z_1 = 0\}$.

2) Observe that $\varphi(z) = u \circ L(z)$, *where*

$$L = z \in (\mathbb{C}^*)^n \mapsto (\log|z_1|, \ldots, \log|z_n|) \in \mathbb{R}^n$$

and u *is an appropriate convex function. Express* $(dd^c\varphi)^n$ *in terms of the real Monge-Ampère measure of* u *and give an alternative proof of the fact that* $(dd^c\varphi)^n$ *has infinite mass near* $\{z_1 = 0\}$.

EXERCISE 3.9. *Fix* $r := (r_1, r_2, \cdots, r_n) \in]0, +\infty[^n$ *and consider*

$$\psi_r : z \in \mathbb{C}^n \mapsto \max\{\log^+(|z_j|/r_j); 1 \leq j \leq n\} \in \mathbb{R}.$$

1) Prove that ψ_r *is a Lipschitz continuous plurisubharmonic function and* $(dd^c\psi_r)^n$ *is supported on the torus*

$$\mathbb{T}^n(r) := \{z \in \mathbb{C}^n; |z_1| = r_1, \cdots, |z_n| = r_n\}.$$

2) Observe that $(dd^c\psi_r)^n$ *is* $(S^1)^n$*-invariant and conclude that* $(dd^c\psi_r)^n$ *is the normalized Lebesgue measure on* $\mathbb{T}^n(r)$.

EXERCISE 3.10. *Set*

$$\varphi(z) := \max_{1 \leq j \leq n} \log^+|z_j| \text{ where } \log^+ x := \max(\log x, 0),$$

and $\varphi_j(z) = \frac{1}{2j}\max(0, \log|z_1^j + \cdots + z_n^j|)$.

1) Show that $\varphi_j \longrightarrow \varphi$ *in* $L^1_{loc}(\mathbb{C}^n)$.

2) Show that $(dd^c\varphi_j)^n = 0$ *when* $n \geq 2$, *while* $(dd^c\varphi)^n$ *is the normalized Lebesgue measure on the torus* $(S^1)^n \subset \mathbb{C}^n$. *Conclude that the complex Monge-Ampère operator is not continuous for the* L^1*-topology.*

EXERCISE 3.11. *Let* φ *be a plurisubharmonic function in* $\mathbb{C}^n$ *whose gradient is in* L^2_{loc}. *Let* φ_j *be a sequence of plurisubharmonic functions decreasing to* φ. *Show that* $\nabla\varphi_j \in L^2_{loc}$ *and that* $\varphi_j \to \varphi$ *in the Sobolev sense* $W^{1,2}_{loc}$.

EXERCISE 3.12. *Let* $u_1, \ldots, u_n$ *be continuous non-negative plurisubharmonic functions in* $\mathbb{C}^n$ *such that for all* i, u_i *is pluriharmonic in* $(u_i > 0)$. *Show that*

$$(dd^c \max(u_1, \ldots, u_n))^n = dd^c u_1 \wedge \cdots \wedge dd^c u_n.$$

EXERCISE 3.13. *Let u be a plurisubharmonic function in $\mathbb{C}^n$. Show that if*

$$\limsup_{|z|\to+\infty} \frac{u(z)}{\log|z|} = 0$$

then u is constant.

EXERCISE 3.14. *Show that if u belongs to the Lelong class $\mathcal{L}^+(\mathbb{C}^n)$, i.e. if there exists a constant $C_u \in \mathbb{R}$ such that for all $z \in \mathbb{C}^n$,*

$$\log^+ ||z|| - C_u \leq u(z) \leq \log^+ ||z|| + C_u,$$

then $\int_{\mathbb{C}^n} (dd^c u)^n = 1$.

EXERCISE 3.15. *Set $u(z) := \log|z|$, $v(z) = \max_{1\leq j\leq n} \log|z_j|$, and*

$$w(z) = \max(\log|z_1|, \log|z_2 - z_1^2|, \ldots, \log|z_n - z_{n-1}^2|).$$

Show that

$$(dd^c u)^n = (dd^c v)^n = (dd^c w)^n = \delta_0$$

is the Dirac mass at the origin of $\mathbb{C}^n$. Conclude that the comparison principle can not hold for these functions.

EXERCISE 3.16. *Set $\varphi(z) = \log^+|z| = \max(\log|z|, 0)$.*

1) Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth convex function and set $\psi(z) = \chi(\log|z|)$. Show that ψ is a psh function with compact singularities. Prove that $(dd^c\psi)^n$ is absolutely continuous with respect to Lebesgue measure if ψ has zero Lelong number at the origin.

2) Approximate $\max(x, 0)$ by a smooth decreasing family of convex functions χ_ε, use 1) and let ε decrease to zero to conclude that $(dd^c\varphi)^n$ is the normalized Lebesgue measure on the unit sphere.

3) Use the invariance properties of φ and Exercise 3.14 to give an alternative proof of this result.

EXERCISE 3.17. *Let u be a smooth plurisubharmonic function in some domain $\Omega \subset \mathbb{C}^n$.*

1) Assume that for all $p \in \Omega$ there exists a holomorphic disc $\mathbb{D} \subset \Omega$ centered at p such that $u_{|\mathbb{D}}$ is harmonic. Prove that $(dd^c u)^n \equiv 0$.

2) Show conversely that if $(dd^c u)^n \equiv 0$ and $(dd^c u)^{n-1} \not\equiv 0$ then there exists a holomorphic foliation of Ω by Riemann surfaces L_a such that $u_{|L_a}$ is harmonic for all a (see [**BK77**] *for some help).*

EXERCISE 3.18. *Let μ be a probability measure in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ and set*

$$V_\mu(z) := \int_{w\in\mathbb{B}} \log|z - w| d\mu(w).$$

Check that V_μ is plurisubharmonic. Give conditions on μ to insure that V_μ is locally bounded, and check that in this case $(dd^c V_\mu)^n$ is absolutely continuous with respect to Lebesgue measure (see [**?**] *for more information).*

CHAPTER 4

The Dirichlet problem

1. Introduction

Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$, $\beta := dd^c|z|^2$ the usual euclidean metric on $\mathbb{C}^n$. Given $\varphi \in C^0(\partial\Omega)$ and $0 \leq f \in C^0(\bar{\Omega})$, we consider the Dirichlet problem for the complex Monge-Ampère operator:

$$(1.1) \qquad \begin{cases} u \in PSH(\Omega) \cap C^0(\bar{\Omega}) & \\ (dd^c u)^n = f\beta^n & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Observe that up to a constant β^n is the volume form on $\mathbb{C}^n$, hnce the right hand side $\mu := f\beta^n$ can be seen as a positive Radon measure on Ω, hence a distribution, so that the equality above must be understood in the weak sens of distribution on Ω.

When $n = 1$ this is the classical Dirichlet problem for the Laplace operator. In this case one can find an explicit formula for the solution, when the domain is sufficiently regular, generalizing the Poisson-Jensen formula in the unit disc proved in Chapter 2.

For more general domains, as well as for the higher dimensional setting, one uses the method of upper-envelopes due to Perron and the comparison principle to build the solution and show it is unique. Namely we consider

$$(1.2) \qquad U(z) = U_{\Omega,\varphi,f}(z) := \sup\{u(z); u \in \mathcal{B}(\Omega, \varphi, f)\},$$

where $\mathcal{B} = \mathcal{B}(\Omega, \varphi, f)$ is the class of subsolutions,

$$\mathcal{B} := \{u \in PSH \cap L^\infty(\Omega); (dd^c u)^n \geq f\beta^n \text{ in } \Omega \text{ and } u^* \leq \varphi \text{ on } \partial\Omega\}.$$

Our goal is to show that when Ω is strictly pseudoconvex domain, then the maximal subsolution U is the unique solution to the Dirichlet problem (1.1).

Recall that a bounded domain $\Omega \Subset \mathbb{C}^n$ is strictly pseudoconvex if it admits a strictly plurisubharmonic defining function i.e. there exists a continuous plurisubharmonic function ρ in Ω t

$$\lim_{z \to \partial\Omega} \rho(z) = 0; \; dd^c\rho \geq \beta, \text{ weakly on } \Omega.$$

A typical example is the euclidean ball in $\mathbb{C}^n$ given by

$$\mathbb{B}_n := \{z \in \mathbb{C}^n; \rho(z) := |z|^2 - 1 < 0\}.$$

When $f = 0$, Bremermann [**Bre59**] has shown that the envelope U is plurisubharmonic and has the right boundary values, by constructing appropriate barriers (this is where the strong pseudoconvexity assumption is used). Later on Walsh [**Wa68**] proved that U is continuous up to the boundary. Then Bedford and Taylor showed in [**BT76**] that the complex Monge-Ampère measure $(dd^c U)^n$ is well defined, and U solves the Dirichlet problem (1.1).

We give in this chapter a complete proof of these results using simplifications due to Demailly in the homogeneous case [**Dem91**], as well as ideas from optimal control and viscosity theory developed recently in [**EGZ11**]. These methods allow a new description of the Perron-Bremermann envelope using Laplace operators associated to a family of constant Kähler metrics on Ω, following an observation by Gaveau [**Gav77**]. This avoids the general measure-theoretic construction of Goffman and Serrin [**?**] used in [**BT76**].

2. The Dirichlet problem for the Laplace operator

We have shown in Chapter 2 that the Poisson transform solves explicitly the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$ (complex dimension $n = 1$). We study here the Laplace operator in $\mathbb{R}^N$ with $N \geq 3$. This will help us later on in getting some information on plurisubharmonic function in domains of $\mathbb{C}^n$, with $N = 2n$.

2.1. The maximum principle. We fix here an integer $N \geq 3$ and a domain $\Omega \subset \mathbb{R}^N$. Let $x = (x_1, .., x_N)$ denote the canonical coordinates in $\mathbb{R}^N$. For $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$ we set

$$x \cdot y := \sum_{j=1}^{N} x_j y_j.$$

and $|x|^2 = x \cdot x = \sum_{j=1}^N x_j^2$. The Laplace operator in $\mathbb{R}^N$ is defined by

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

Harmonic functions in a domain $\Omega \subset \mathbb{R}^N$ are those which satisfy the Laplace equation $\Delta u = 0$ in Ω (in the smooth or weak sense of distributions). They can also be characterized as continuous functions which satisfy the spherical mean-value property.

DEFINITION 4.1. *A function $u : \Omega \subset\longrightarrow [-\infty, +\infty[$ is subharmonic if it is upper semi-continuous, and for any ball $\mathbb{B}(a, r) \Subset \Omega$,*

$$u(a) \leq \frac{1}{\sigma_{N-1}} \int_{|\xi|=1} u(+a + r\xi) d\sigma(\xi),$$

where $d\sigma$ is the area measure on the unit sphere $\mathbb{S} := \{x \in \mathbb{R}^N ; |x| = 1\}$.

It follows from Chapter 2 that plurisubharmonic functions in a domain $\Omega \subset \mathbb{C}^n$ are subharmonic as functions of $2n$-real variables.

We denote by $SH(\Omega)$ the positive cone of subharmonic functions in the domain Ω which are not identically $-\infty$. It follows from the submean value inequalities that

$$SH(\Omega) \subset L^1_{loc}(\Omega).$$

PROPOSITION 4.2. *Let u be a subharmonic function in a bounded domain $\Omega \Subset \mathbb{R}^N$ such that* $\limsup_{x\to\partial\Omega} u(x) \leq 0$. *Then $u \leq 0$ in Ω. Moreover for all $x \in \Omega$,*

$$u(x) < \sup_{\partial\Omega} u$$

unless u is constant in Ω.

The proof of this maximum principle is similar to the one given in Chapter 2.

2.2. Green functions. For $N \geq 3$, the Newton Kernel

$$K_N(x) := \frac{-1}{(N-2)\sigma_{N-1}} |x|^{2-N}$$

is a locally integrable function in $\mathbb{R}^N$ which satisfies

$$\Delta K_N = \delta_0$$

in the sense of distribution in $\mathbb{R}^N$, where δ_0 is the Dirac measure at the origin. In particular, K_N is subharmonic in $\mathbb{R}^N$ and harmonic in $\mathbb{R}^N \setminus \{0\}$. It is the *fundamental solution* to the Laplace operator in $\mathbb{R}^N$.

DEFINITION 4.3. *The Green function for the unit ball $\mathbb{B}$ is*

$$G(x,y) = G_{\mathbb{B}}(x,y) := K_N(x-y) - K_N(|y|x - y/|y|),$$

where $(x,y) \in \mathbb{B} \times \mathbb{B}$.

Observe that G is well defined in $\mathbb{B} \times \mathbb{B}$ and has the same singularities as $K_N(x-y)$ since the second term is smooth. It satisfies the following properties:

(1) For all $y \in \mathbb{B}$, $x \longmapsto G(x,y)$ is subharmonic in $\mathbb{B}$,
(2) $G(x,y) = G(y,x)$ in $\mathbb{B} \times \mathbb{B}$,
(3) $G < 0$ in $\mathbb{B} \times \mathbb{B}$, and $G(x,y) = 0$ if $(x,y) \in \partial\mathbb{B} \times \mathbb{B}$,
(4) for all $y \in \mathbb{B}$, $\Delta_x G(x,y) = \delta_y$.

DEFINITION 4.4. *The function $x \longmapsto G(x,y)$ is called the Green function of the ball $\mathbb{B}$ with pole at y.*

It follows from the maximum principle for the Laplace operator that G is unique function satisfying these properties.

The explicit formula for the Newtonian Kernel K_N yields

$$G(x,y) = \frac{-1}{(N-2)\sigma_{N-1}} \left(\| x - y|^{2-N} - (1 - 2x\cdot y + |x|^2|y|^2)^{1-N/2} \right).$$

Recall that if u, v are smooth functions in $\mathbb{B}$, then

$$\int_{\mathbb{B}} (u\Delta v - v\Delta u) d\lambda = \int_{\partial\mathbb{B}} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{v\partial \nu} \right) d\sigma.$$

Here $\partial/\partial\nu$ is the derivative along the outward normal vector ν in $\partial\mathbb{B}$ and $d\sigma$ is the euclidean area measure on $\partial\mathbb{B} = \mathbb{S}$. To get a representation formula for a subharmonic function u we apply this formula with $v := G(x, \cdot)$ and $x \in \mathbb{B}$ fixed. We thus consider the Poisson kernel

$$P(x, y) := \partial G(x, y)/\partial\nu(y),\ (x, y) \in \mathbb{B} \times \partial\mathbb{B}.$$

An easy computation shows that

$$P(x, y) := \frac{1}{(N-2)\sigma_{N-1}} \frac{1 - |x|^2}{|x - y|^N}, (x, y) \in \mathbb{B} \times \partial\mathbb{B}.$$

Proposition 4.5. *Let $u \in SH(\mathbb{B}) \cap C^0(\bar{\mathbb{B}})$. Then for all $x \in \mathbb{B}$,*

$$u(x) = \int_{\mathbb{S}} u(y)P(x, y)d\sigma(y) + \int_{\mathbb{B}} G(x, y)d\mu_u(y), \tag{2.1}$$

where $\mu_u := \Delta u$ is the the Riesz measure of u.

In particular if u is harmonic in $\mathbb{B}$ and continuous in $\bar{\mathbb{B}}$ then

$$u(x) = \int_{\mathbb{S}} u(y)P(x, y)d\sigma(y). \tag{2.2}$$

The proof is left as an Exercise 4.2. We can thus solve the Dirichlet problem for the homogeneous Laplace equation with continuous boundary values:

Theorem 4.6. *Fix $\varphi \in \mathcal{C}^0(\partial\mathbb{B})$. The Poisson transform of φ,*

$$P_\varphi(x) := \int_{|y|=1} \varphi(y)P(x, y)d\sigma(y),\ x \in \mathbb{B}, \tag{2.3}$$

is harmonic in $\mathbb{B}$, continuous in $\bar{\mathbb{B}}$ and satisfies $P_\varphi(x) = \varphi(x)$ for $x \in \mathbb{S}$.

The proof is identical to the one for the unit disc (see Chapter 2).

2.3. A characterization of subharmonic functions. The following characterization of subharmonic functions will be quite useful:

Proposition 4.7. *Let $u : \Omega \longrightarrow \mathbb{R}^N$ be an upper semi-continuous function in a domain $\Omega \subset \mathbb{R}^N$. The following conditions are equivalent:*

1. the function u is subharmonic in Ω;

2. $\Delta q(x_0) \geq 0$ for all $x_0 \in \Omega$ and all C^2-smooth functions q in a small ball B of center x_0 such that $u \leq q$ in B and $u(z_0) = q(z_0)$;

3. $u \leq h$ in B, for all balls $B \Subset \Omega$ and all functions $h : \bar{B} \longrightarrow \mathbb{R}$ continuous in $\bar{B}$ and harmonic in B such that $u \leq h$ in ∂B.

A C^2-smooth function q in a ball $B = B(x_0, r)$ s.t. $u \leq q$ in B and $u(x_0) = q(x_0)$ is called an upper test function for u at the point x_0.

PROOF. We first prove (1) $\Longrightarrow$ (2). Assume that u is subharmonic and fix $x_0 \in \Omega$. If there exists a C^2-smooth upper test function q at x_0 such that $\Delta q(x_0) < 0$, then for $\varepsilon > 0$ small enough the function $x \longmapsto u(x) - q(x) - \varepsilon|x - x_0|^2$ is subharmonic in a small ball $B(x_0, r)$, equal to 0 at x_0 and negative for $0 < |x - x_0| < r$ small enough. This contradicts the maximum principle.

We now prove (2) $\Longrightarrow$ (3). Let h be a harmonic function in some $B(a, r) \Subset \Omega$, continuous on $\bar{B}(a, r)$, and such that $u \le h$ in $\partial B(a, r)$. Fix $\varepsilon > 0$ and observe that

$$u(x) \le h_\varepsilon(x) := h(x) - \varepsilon(|x - a|^2 - r^2) \text{ on } \partial B(a, r).$$

By upper semi-continuity, there exists $x_0 \in \bar{B}(a, r)$ such that

$$u(x_0) - h_\varepsilon(x_0) = \max_{\bar{B}(a,r)} (u - h_\varepsilon)$$

If $x_0 \in B$, we can fix $B(z_0, s) \subset B(a, r)$ so that $q := h_\varepsilon - h_\varepsilon(x_0) + u(x_0)$ is a smooth upper test function for u at x_0 s.t. $\Delta q(x_0) = -2n\varepsilon < 0$, a contradiction. Thus $x_0 \in \partial B$ and $u \le h_\varepsilon$ in $B(a, r)$ for all $\varepsilon > 0$. We infer $u \le h$ in $B(a, r)$ by letting $\varepsilon \to 0$.

We finally prove (3) $\Longrightarrow$ (1). Fix $a \in \Omega$ and $r > 0$ such that $\bar{B}(a, r) \subset \Omega$. Let (φ_j) be a sequence of continuous functions decreasing to u in ∂B. It follows from Theorem 4.6 that there exists a harmonic function h_j in B continuous in $\bar{B}$ such that $h_j = \varphi_j$ in ∂B. Property (3) yields $u \le h_j$ in B for any $j \ge 1$. Thus

$$u(a) \le h_j(a) = \frac{1}{\sigma_{N-1}} \int_{|y|=1} \varphi_j(a + ry) d\sigma(y).$$

We infer ($j \to +\infty$) that u satisfies the submean value inequalities. $\square$

The implication (1) $\Longrightarrow$ (2) is a soft version of the maximum principle. Property (2) can be used to define plurisubharmonicity in the sense of viscosity as we explain in the sequel.

3. The Perron-Bremermann envelope

3.1. A characterization of plurisubharmonicity. Let H_n^+ denote the set of all semi-positive hermitian $n \times n$ matrices. We set

$$\dot{H}_n^+ := \{H \in H_n^+ ; \det H = n^{-n}\}.$$

The following observation of Gaveau [**Gav77**] is quite useful:

LEMMA 4.8. *Fix $Q \in H_n^+$. Then*

$$(det\ Q)^{\frac{1}{n}} = \inf\{\mathrm{tr}(H\, Q) \ ; \ H \in \dot{H}_n^+\}$$

PROOF. Every matrix $H \in \dot{H}_n^+$ has a square root which we denote by $H^{1/2} \in H_n^+$. Thus $H^{1/2} \cdot Q \cdot H^{1/2} \in H_n^+$. Diagonalizing the latter and using the arithmetico-geometric inequality, we get

$$(\det\ Q)^{\frac{1}{n}} (\det H)^{\frac{1}{n}} = (\det(H^{1/2} \cdot Q \cdot H^{1/2}))^{\frac{1}{n}} \le \frac{1}{n} \mathrm{tr}(H^{1/2} \cdot Q \cdot H^{1/2}).$$

Therefore $(\det Q)^{\frac{1}{n}}(\det H)^{\frac{1}{n}} \leq \frac{1}{n} tr(Q.H)$, hence

$$(\det Q)^{\frac{1}{n}} \leq \inf\{tr(H.Q); H \in \dot{H}_n^+\}.$$

Suppose now that $Q \in H_n^+$ is positive. There exists P an invertible hermitian matrix and a diagonal matrix $A = (\lambda_i)$ with positive entries $\lambda_i > 0$ such that $Q = P.A.P^{-1}$. Set

$$\alpha_i = \frac{(\prod_i \lambda_i)^{\frac{1}{n}}}{n\lambda_i}.$$

Observe that $H = (\alpha_i) \in \dot{H}_n^+$ and $(\det A)^{\frac{1}{n}} = tr(A.H)$, hence

$$(\det Q)^{\frac{1}{n}} = (\det A)^{\frac{1}{n}} = tr(H \cdot A) = tr(H \cdot P \cdot A \cdot P^{-1}) = tr(H' \cdot Q),$$

where $H' = P \cdot H \cdot P^{-1} \in \dot{H}_n^+$.

If Q is merely semi-positive we consider $Q_\varepsilon := Q + \varepsilon I_n$, $\varepsilon > 0$, and apply the previous argument to obtain

$$(\det Q_\varepsilon)^{\frac{1}{n}} = \inf\{tr(H.Q_\varepsilon); H \in \dot{H}_n^+\} \geq \inf\{tr(H.Q); \dot{H}_n^+\}.$$

We conclude by letting $\varepsilon \to 0$. □

For $H \in \dot{H}_n^+$, we consider

$$\Delta_H := \sum_{j,k=1}^{n} h_{k\bar{j}} \frac{\partial^2}{\partial z_j \partial \bar{z}_k}, \tag{3.1}$$

the Laplace operator associated to the (constant) Kähler metric defined by $\bar{H}^{-1}$ in $\mathbb{C}^n$. The previous lemma yields the following interesting characterization of plurisubharmonicity:

PROPOSITION 4.9. *Let $u : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semi-continuous function. The following properties are equivalent*

(i) The function u is plurisubharmonic in Ω;

(ii) $dd^c q(z_0) \geq 0$ for all $z_0 \in \Omega$ and all functions q C^2 in a neighborhood B of z_0 such that $u \leq q$ in B and $u(z_0) = q(z_0)$;

(iii) u is subharmonic and for all $H \in \dot{H}_n^+$, $\Delta_H u \geq 0$ in the sense of distributions.

PROOF. We first prove $(i) \Longrightarrow (ii)$. Assume that is u is plurisubharmonic and fix $z_0 \in \Omega$. If there exists a C^2-smooth upper test q at z_0 such that $dd^c q(z_0)$ is not semi-positive, there is a direction $\xi \in \mathbb{C}^n$, $\xi \neq 0$ such that $\sum_{j,k} \xi_j \bar{\xi}_k \frac{\partial^2 q}{\partial z_j \partial \bar{z}_k}(z_0) < 0$. Then for $\varepsilon > 0$ small enough

$$\tau \longmapsto u(z_0 + \tau\xi) - q(z_0 + \tau\xi) - \varepsilon|\tau|^2$$

is subharmonic in a neighborhood of the origin where it reaches a local maximum, contradicting the maximum principle.

We now prove $(ii) \Longrightarrow (iii)$. Let h be a function that is harmonic in a ball $B = B(a,r) \Subset \Omega$, continuous on $\bar{B}(a,r)$, and such that $u \leq h$ in $\partial B(a,r)$. We claim that $u \leq h$ in B. Fix $\varepsilon > 0$ and observe that

$$u(x) \leq h_\varepsilon(x) := h(x) - \varepsilon(|x-a|^2 - r^2) \text{ on } \partial B(a,r).$$

By upper semi-continuity, there exists $x_0 \in \bar{B}(a,r)$ such that

$$u(x_0) - h_\varepsilon(x_0) = \max_{\bar{B}(a,r)} (u - h_\varepsilon).$$

If $x_0 \in B$, we can choose a small ball $B(x_0, s) \subset B$ so that

$$q := h_\varepsilon - h_\varepsilon(x_0) + u(x_0)$$

is an upper test for u at x_0 which satifies $\Delta q(x_0) = -2n\varepsilon < 0$, a contradiction. Therefore $x_0 \in \partial B$ and $u \leq h_\varepsilon$ in $B(a,r)$ for all $\varepsilon > 0$. We infer $u \leq h$ in $B(a,r)$ as $\varepsilon \to 0$.

This proves that u is subharmonic in Ω by Proposition 4.7, hence $\Delta u \geq 0$ in Ω in the sense of distributions. To prove that $\Delta_H u \geq 0$ for all $H \in \dot{H}_n^+$, we use the following observations:

(a) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}$-linear isomorphism and $q : \mathbb{C}^n_\zeta \to \mathbb{C}^n_z$ be a C^2-smooth function in a neighborhood of a point z_0. Set $z = T(\zeta)$ and $q_T(\zeta) := q(z) = q(T(\zeta))$, then q_T is C^2-smooth function in a neighborhood of $\zeta_0 := T^{-1}(z_0)$ and

$$\Delta q_T(\zeta) = \sum_{j=1}^{n} \frac{\partial^2 q_T(\zeta)}{\partial \zeta_j \partial \bar{\zeta}_k} = \operatorname{tr}(T^* Q(z) T),$$

where T^* denotes the complex conjugate transpose of $T := (\frac{\partial z_j}{\partial \zeta_k})$ and $Q(z) := (\frac{\partial^2 q}{\partial z_j \partial \bar{z}_k}(z))$ is the complex hessian of q at z. If $H \in \dot{H}_n^+$ we can find a hermitian positive matrix T s.t. $T^* T = H$, one then gets

$$\Delta q_T(\zeta) = \Delta_H q(z).$$

We leave the details as an Exercise 4.4.

(b) Fix $u : \Omega \longrightarrow [-\infty, +\infty[$ an upper semi-continuous function and $z_0 \in \Omega$. Then q is an upper test function for u at z_0 iff $\tilde{q} := q_T := q \circ T$ is an upper test function for u_T at the point $\zeta_0 := T^{-1}(z_0)$.

Observation (b) shows that the condition (ii) is invariant under complex linear change of coordinates. Observation (a) and the proof preceeding it show that (ii) implies (iii).

We finally show that $(iii) \Longrightarrow (i)$. Assume first that u is smooth. Condition (iii) means that the complex hessian matrix A of u at $z_0 \in \Omega$ is a hermitian matrix that satisfies $\operatorname{tr}(HA) \geq 0$ for all $H \in H_n$. We infer, by diagonalizing A, that A is a semi-positive hermitian matrix. Thus u is plurisubharmonic in Ω.

To treat the general case we regularize $u_\varepsilon = u \star \rho_\varepsilon$ (see Chapter 1) and obtain smooth functions satisfying the Condition (iii) since

$$\Delta_H u_\varepsilon = (\Delta_H u) \star \rho_\varepsilon \text{ in } \Omega_\varepsilon$$

by linearity. Thus u_ε is plurisubharmonic in Ω_ε. Since u is subharmonic in Ω, it follows that u_ε decreases to u as ε decreases to 0 hence u is plurisubharmonic in Ω. □

3.2. Perron envelopes.

3.2.1. *The viscosity point of view.* We establish here some useful facts which are also the first steps towards a viscosity approach to solving complex Monge-Ampère equations (see [**GZ17**]).

Proposition 4.10. *Let* $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ *and* $0 \leq f \in C^0(\Omega)$. *The following conditions are equivalent :*

(1) $(dd^c u)^n \geq f\beta^n$;

(2) $\Delta_H u \geq f^{1/n}$ *for all* $H \in \dot{H}_n^+$.

This equivalence has been observed by Blocki [**Blo96**, Theorem 3.10] when u continuous, by using a slightly different argument.

Proof. We first prove (2) $\Longrightarrow$ (1). Suppose that $u \in C^2(\Omega)$. It follows from Lemma 4.8 that

$$\Delta_H u \geq f^{1/n}, \forall H \in \dot{H}_n^+$$

is equivalent to

$$\left(det(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}) \right)^{1/n} \geq f^{1/n},$$

which is itself equivalent to $(dd^c u)^n \geq f\beta^n$. All these inequalities hold pointwise in Ω.

When u is not smooth, we fix $H \in \dot{H}_n^+$ and let (χ_ϵ) be standard mollifiers. The functions $u_\epsilon := u \star \chi_\epsilon$ are plurisubharmonic in Ω_ε and decrease to u as ε decreases to 0. Since the conditions (2) are linear we infer $\Delta_H u_\epsilon \geq (f^{1/n})_\epsilon$ pointwise in Ω_ε. We can use the first case since u_ϵ is smooth, obtaining $(dd^c u_\epsilon)^n \geq ((f^{1/n})_\epsilon)^n \beta^n$ pointwise in Ω_ε. Letting $\epsilon \searrow$ and applying the convergence theorem for the complex Monge-Ampère operator, we obtain $(dd^c u)^n \geq f\beta^n$ weakly in Ω.

We now prove that (1) $\Longrightarrow$ (2). Fix $x_0 \in \Omega$ and q a C^2-smooth function in a neighborhood B of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$. We know from Proposition 4.9 that $dd^c q(x_0) \geq 0$. We claim that $(dd^c q(x_0))^n \geq f(x_0)\beta^n$. Suppose by contradiction that $(dd^c q)^n_{x_0} < f(x_0)\beta^n$ and set

$$q^\epsilon(x) = q(x) + \epsilon \left(\|x - x_0\|^2 - \frac{r^2}{2} \right).$$

If $0 < \epsilon < 1$ is small then $0 < (dd^c q^\epsilon(x_0))^n < f(x_0)\beta^n$. Since f is lower semi-continuous at x_0, there exists $r > 0$ such that

$$(dd^c q^\epsilon(x))^n \leq f(x)\beta^n \quad \text{in} \quad \mathbb{B}(x_0, r).$$

It follows that $(dd^c q^\epsilon)^n \leq (dd^c u)^n$ in $\mathbb{B}(x_0, r)$ and $q^\epsilon \geq q \geq u$ on $\partial\mathbb{B}(x_0, r)$. The comparison principle implies that $q^\epsilon \geq u$ on $\mathbb{B}(x_0, r)$. But $q^\epsilon(x_0) = q(x_0) - \epsilon\frac{r^2}{2} = u(x_0) - \epsilon\frac{r^2}{2} < u(x_0)$, a contradiction.

It follows from Proposition 4.9 that for any $x_0 \in \Omega$, and every upper test q for u at x_0, we have $\Delta_H q(x_0) \geq f^{1/n}(x_0)$ for any $H \in \dot{H}_n^+$.

If f is positive and smooth, there exists $g \in C^\infty(\bar{\Omega})$ such that $\Delta_H g = f^{1/n}$. Thus $h = u - g$ is Δ_H-subharmonic by Proposition 4.7, i.e. $\Delta_H h \geq 0$, hence $\Delta_H u \geq f^{1/n}$ for any $H \in \dot{H}_n^+$.

If f is positive and merely continuous, we observe that

$$f = \sup\{g;\ g \in C^\infty(\bar{\Omega}), f \geq g > 0\},$$

hence $(dd^c u)^n \geq f\beta^n \geq g\beta^n$ for any such g. The previous case yields $\Delta_H u \geq g^{1/n}$. We infer $\Delta_H u \geq f^{1/n}$ for all $H \in \dot{H}_n^+$.

Assume finally f is merely continuous and semipositive. Observe that $u^\epsilon(z) = u(z) + \epsilon\|z\|^2$ satisfies

$$(dd^c u^\epsilon)^n \geq (f + \epsilon^n)\beta^n.$$

It follows from the previous case that for all $H \in \dot{H}_n^+$,

$$\Delta_H u^\epsilon \geq (f + \epsilon^n)^{1/n}.$$

Letting ϵ decrease to 0 yields $\Delta_H u \geq f^{1/n}$ for all $H \in \dot{H}_n^+$. □

3.2.2. *Perron-Bremermann envelopes.* Previous proposition allows us to reinterpret the Perron-Bremermann envelope by using the Laplace operators Δ_H. Consider

$$\mathcal{V} = \{v \in PSH \cap L^\infty(\Omega), v|_{\partial\Omega} \leq \varphi \textit{ and } \Delta_H v \geq f^{1/n}\ \forall H \in \dot{H}_n^+\}.$$

PROPOSITION 4.11. *The class $\mathcal{V}$ is non-empty, stable under maxima and bounded from above in Ω. The Perron-Bremermann envelope*

$$U_{\Omega,\varphi,f}(z) = \sup\{v(z); v \in \mathcal{V}\} \tag{3.2}$$

is plurisubharmonic in Ω.

PROOF. Let ρ be a strictly plurisubharmonic defining function for Ω. Choose $A > 0$ big enough so that $Add^c\rho \geq M^{1/n}\beta$, where $M := \|f\|_{L^\infty(\Omega)}$. We use here that Ω is bounded and ρ is strictly plurisubharmonic near Ω. Fix $B > 0$ so large that $-B \leq \varphi \leq B$. Then $v_0 := A\rho - B \in \mathcal{V}$ since

$$\Delta_H v_0 \geq A\Delta_H\rho \geq M^{1/n} \geq f^{1/n},$$

for all $H \in \dot{H}_n^+$. Thus $\mathcal{V} \neq \emptyset$.

Since φ is bounded from above by B, the maximum principle shows that all functions in $\mathcal{V}$ are bounded from above by B. It follows that $U := U_{\Omega,\varphi,f}$ is well defined and given by

$$U(z) = \sup\{v(z); v \in \mathcal{V}_0\},\ z \in \Omega,$$

where

$$\mathcal{V}_0 := \{v \in \mathcal{V}; v_0 \leq v \leq M\}.$$

We claim that $\mathcal{V}_0 \subset L^1(\Omega)$ is compact. Indeed let (v_j) be a sequence in $\mathcal{V}_0$. This is a bounded sequence of plurisubharmonic functions in $L^1(\Omega)$. Thus there exists a subsequence (w_j) such that $w_j \to w$ in $L^1(\Omega)$, where w is plurisubharmonic and satisfies $w = (\limsup_j w_j)^*$ in Ω. We infer $v_0 \leq w \leq B$ and for all $H \in \dot{H}_n^+$,

$$f^{1/n} \leq \Delta_H w_j \to \Delta_H w,$$

hence $\Delta_H w \geq f^{1/n}$. Therefore $w \in \mathcal{V}_0$, which proves the claim. It follows that U is plurisubharmonic in Ω.

We now prove that the class $\mathcal{V}$ is stable under maxima, that is if $u, v \in \mathcal{V}$ then $\max\{u, v\} \in \mathcal{V}$. It suffices to show that for all $H \in \dot{H}_n^+$

$$\Delta_H \max\{u, v\} \geq \min(\Delta_H u, \Delta_H v) \tag{3.3}$$

Indeed let $\mu := \min\{\Delta_H u, \Delta_H v\}$ in the sense of Radon measures in Ω and suppose that $\mu(\{z; u(z) = v(z)\}) = 0$. The local maximum principle shows that $\Delta_H \max\{u, v\} \geq \mu$ in the sense of Borel measures in the Borel set $\Omega' = \{u \neq v\}$. Since $\mu(\Omega \setminus \Omega') = 0$, we get

$$\Delta_H \max\{u, v\} \geq \mu := \min\{\Delta_H u, \Delta_H v\}.$$

When $\mu(\{z; u(z) = v(z)\}) \neq 0$ we replace v by $v + \epsilon$, and observe that $\mu(\{z; u(z) = v(z) + \epsilon\}) \neq 0$ for at most countably many ϵ's. The previous case yields $\Delta_H \max\{u, v + \epsilon\} \geq \min\{\Delta_H u, \Delta_H v\} = \mu$ for those ε's. Since $\Delta_H \max\{u, v + \epsilon\}$ converges to $\Delta_H \max\{u, v\}$, we obtain (3.3). □

3.3. Continuity of the envelope.

THEOREM 4.12. *Let $0 \leq f \in C(\bar{\Omega})$ be a continuous function in $\bar{\Omega}$ and $\varphi \in C(\partial\Omega)$. The Perron-Bremermann envelope*

$$U = sup\{v;\ v \in \mathcal{V}(\Omega, \varphi, f)\}$$

is a continuous plurisubharmonic function which belongs to $\mathcal{V}(\Omega, \varphi, f)$ and satisfies $U = \varphi$ on $\partial\Omega$.

If $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ then the modulus of continuity of U satisfies

$$\omega_U(\delta) \leq C\delta \|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)} + B\omega_{f^{1/n}}(\delta),$$

where B, C only depend on the geometry of the domain Ω. In particular U is Lipschitz on $\bar{\Omega}$ whenever $f^{1/n}$ is Lipschitz on $\bar{\Omega}$.

PROOF. The proof proceeds in several steps.

Step 1. We first show that $U \in \mathcal{V}$. We have already shown in Proposition 4.11 that U is plurisubharmonic and bounded. It follows from Choquet's lemma that there exists a sequence (v_j) in $\mathcal{V}(\Omega, \varphi, f)$ such that

$$U = (\sup_j v_j)^* \text{ in } \Omega.$$

By Proposition 4.11 (stability of the family under max), we can further assume that (v_j) in non decreasing. Fix $H \in \dot{H}_n^+$. Since $\Delta_H v_j \geq f^{1/n}$ for all j and $v_j \to u$ in $L^1(\Omega)$, we infer $\Delta_H u \geq f^{1/n}$, hence $U \in \mathcal{V}$.

Step 2. Contruction of barriers at boundary points. Let ρ be a strictly plurisubharmonic defining function for Ω with $dd^c\rho \geq c_0 dd^c|z|^2$, for some $c_0 > 0$. Fix $\varepsilon > 0$ and let ψ be a $C^{1,1}$-smooth function in $\bar{\Omega}$ such that $|\psi - \varphi| \leq \varepsilon$ on $\partial\Omega$. For $K > 1$ large enough,

$$v_0 := K\rho + \psi - 2\varepsilon,$$

is a continuous function near $\bar{\Omega}$ such that $v_0 \leq \varphi$ on $\partial\Omega$ and

$$dd^c v_0 = K dd^c\rho + dd^c\psi \geq M^{1/n}\beta,$$

where $M = \sup_\Omega f$. Observe that K depends on the $C^{1,1}$-bound of ψ. Fix $H \in \dot{H}_n^+$. Then

$$\Delta_H v_0 \geq f^{1/n}.$$

Therefore v_0 belongs to the class $\mathcal{V}$ and $v_0 \leq U$. It follows that

$$\liminf_{z\to\zeta} U(z) \geq \psi(\zeta) - 2\varepsilon,$$

for all $\zeta \in \partial\Omega$. Letting ψ converge to φ and then $\varepsilon \to 0$, we obtain

$$\liminf_{z\to\zeta} U(z) \geq \varphi(\zeta).$$

The same argument shows that

$$w_0 := K\rho - \psi - 2\varepsilon$$

is plurisubharmonic and continuous on $\bar{\Omega}$, with $-\varphi - \varepsilon \leq w_0 \leq -\varphi$. Observe that $U \leq -w_0$. Indeed if $v \in \mathcal{V}(\Omega, \varphi, f)$ then $v + w_0$ is a bounded plurisubharmonic function in Ω that satisfies $v^* + w_0 \leq 0$ on $\partial\Omega$. The maximum principle yields $v + w_0 \leq 0$ hence $U \leq -w_0$ in Ω. We infer that $\limsup_{z\to\zeta} U(z) \leq \varphi(\zeta)$, for all $\zeta \in \partial\Omega$. Thus

$$\lim_{z\to\zeta} U(z) = \varphi(\zeta). \tag{3.4}$$

Step 3. U is continuous on $\bar{\Omega}$. It follows from the above estimates that for $\zeta \in \partial\Omega$ and $z \in \Omega$,

$$U(z) - \varphi(\zeta) \leq -w_0(z) - \varphi(\zeta) \leq -K\rho(z) + \varepsilon. \tag{3.5}$$

Fix $C > 0$ such that $-\rho(z) \leq C|z - \zeta|$ for $\zeta \in \partial\Omega$ and $z \in \Omega$. If $\delta > 0$ is small enough we infer, for $z \in \Omega$, $\zeta \in \partial\Omega$,

$$|z - \zeta| \leq \delta \Longrightarrow U(z) - \varphi(\zeta) \leq KC\delta + \varepsilon.$$

Fix $a \in \mathbb{C}^n$, $|a| < \delta$, and define $\Omega_a := \Omega - a$. For $v \in \mathcal{V}$ we set

$$v_1 := \max\{v, v_0\}.$$

It follows from (3.3) that $v_1 \in \mathcal{V}$ and $\varphi - \varepsilon \leq v_1 \leq \varphi$ in $\partial\Omega$. Moreover if $\zeta \in \Omega_a \cap \partial\Omega$,

$$v_1(\zeta + a) \leq U(\zeta + a) \leq \varphi(\zeta) + KC\delta + \varepsilon,$$

while $v_1(z+a) \leq \varphi(z+a) + KC\delta + \varepsilon$, if $z \in \Omega \cap \partial\Omega_a$. Therefore

$$w(z) = \begin{cases} v_1(z), & \text{if } z \in \Omega \setminus \Omega_a \\ \max\{v_1(z), v_1(z+a) - \varepsilon - KC\delta\} & \text{if } z \in \Omega \cap \Omega_a \end{cases}$$

is a bounded plurisubharmonic function in Ω such that $w \leq \varphi$ in $\partial\Omega$ and, by (3.3), for all $H \in \dot{H}_n^+$,

$$\Delta_H w(z) \geq \min\{f^{1/n}(z), f^{1/n}(z+a)\}$$

in $\Omega \cap \Omega_a$, while $\Delta_H w(z) \geq f^{1/n}(z)$ in $\Omega \setminus \bar{\Omega}_a$.

Let $\omega_{f^{1/n}}$ denote the modulus of continuity of $f^{1/n}$ in Ω. Since $|a| \leq \delta$, it follows that $f^{1/n}(z+a) \geq f^{1/n}(z) - \omega_{f^{1/n}}(\delta)$ hence

$$\Delta_H w(z) \geq f^{1/n}(z) - \omega_{f^{1/n}}(\delta)$$

in Ω. The function

$$\tilde{w}(z) := \omega_{f^{1/n}}(\delta)\rho(z) + w(z),\ z \in \Omega,$$

therefore satisfies $\tilde{w} \in \mathcal{V}$ and $w^* \leq \varphi$ in $\partial\Omega$, hence $\tilde{w} \leq U$ in Ω. Thus

$$v(z+a) \leq v_1(z+a) - \varepsilon - KC\delta \leq U(z) + B\omega_{f^{1/n}}(\delta),$$

if $z \in \Omega$ and $z + a \in \Omega$, where $B = -\inf_\Omega \rho$. Since v was arbitrary in $\mathcal{V}$, it follows that

$$U(z+a) - \varepsilon - KC\delta - B\omega_{f^{1/n}}(\delta) \leq U(z),$$

if $z \in \Omega$ and $z + a \in \Omega$.

This proves that U is continuous in Ω, hence on $\bar{\Omega}$ by (3.4). Therefore $U \in \mathcal{V}$ satisfies the requirements of the first part of the theorem.

Step 4. Modulus of continuity of U. In the construction above the constants B, C do not depend on ε and δ, while the constant $K = K(\psi)$ depends only on the $C^{1,1}$-bound of an ε-approximation ψ of φ in $\partial\Omega$. When φ is $C^{1,1}$-smooth we can take $\psi = \varphi$ and thus get a precise control on the modulus of continuity of U: for $\delta > 0$ small enough,

$$\omega_U(\delta) \leq C\delta||\varphi||_{C^{1,1}} + B\omega_{f^{1/n}}(\delta),$$

as desired. □

4. The case of the unit ball

We are going to show that the Perron-Bremermann envelope U solves the Monge-Ampère equation $(dd^c U)^n = f\beta^n$ in Ω. Following [**BT76**] (and simplifications by [**Dem91**]) we first prove this statement when $\Omega = \mathbb{B}$ is the unit ball and f, φ are regular enough.

4.1. $\mathcal{C}^{1,1}$-regularity.

THEOREM 4.13. *Assume $\Omega = \mathbb{B}$ is the unit ball, $f^{1/n} \in C^{1,1}(\bar{\mathbb{B}})$ and $\varphi \in C^{1,1}(\partial\mathbb{B})$. Then the Perron-Bremermann envelope $U = U_{\mathbb{B},\varphi,f}$ admits second order partial derivates almost everywhere in $\mathbb{B}$ which are locally bounded in $\mathbb{B}$, i.e. $U \in C^{1,1}_{loc}(\mathbb{B})$.*

Here and in the sequel we set

$$C^{0,\alpha}(\overline{\Omega}) = \{v \in C(\overline{\Omega});\ \|v\|_\alpha < +\infty\}$$

for $0 < \alpha \leq 1$, where the α-Hölder norm is given by

$$\|v\|_\alpha = \|v\|_{\alpha,\overline{\Omega}} = \sup\{|v(z)| : z \in \overline{\Omega}\} + \sup\left\{\frac{|v(z) - v(y)|}{|z-y|^\alpha} : z, y \in \overline{\Omega}\right\}.$$

If $0 < \alpha \leq 1$ and $k \in \mathbb{N}^*$, then $C^{k,\alpha}(\overline{\Omega})$ denotes the class of functions which admits continuous partial derivatives up to order k, and whose k-th order partial derivatives are Hölder continuous of order α in $\overline{\Omega}$. We shall also consider the spaces $C^{k,\alpha}_{loc}(\Omega)$ and $C^{k,\alpha}(\partial\Omega)$ with obvious notations.

PROOF. The proof of Theorem 4.13 consists of several steps and occupies the rest of this section. Recall from Theorem 4.12 that $U \in C^{0,1}(\bar{\mathbb{B}})$. We are going to show that for any fixed compact $K \subset \mathbb{B}$, there exists $C = C(K) > 0$ such that for any $z \in K$ and $|h|$ small enough,

$$U(z+h) + U(z-h) - 2U(z) \leq C|h|^2. \tag{4.1}$$

This implies that U has second order partial derivatives almost everywhere that are locally bounded.

Step 1: Using automorphisms of the ball $\mathbb{B}$ as translations. The main difficulty with the expression $U(z+h)+U(z-h)-2U(z)$ is that it is not defined in $\mathbb{B}$ since translations do not preserve the ball. We use automorphisms of $\mathbb{B}$ instead and study the corresponding invariant symmetric differences of second order. For $a \in \mathbb{B}$, we set

$$T_a(z) = \frac{P_a(z) - a + \sqrt{1-|a|^2}(z - P_a(z))}{1 - \langle z, a\rangle} \quad ; \quad P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a$$

where $\langle \cdot, \cdot \rangle$ denote the Hermitian product in $\mathbb{C}^n$. The reader will check in Exercise 4.7 that T_a is a holomorphic automorphism of the unit ball such that $T_a(a) = 0$. Note that T_0 is the identity. We set

$$h = h(a, z) := a - \langle z, a\rangle z. \tag{4.2}$$

Observe that $h(-a, z) = -h(a, z)$. If $|a| \leq 1/2$ then

$$T_a(z) = z - h + O(|a|^2),$$

where $O(|a|^2) \leq C|a|^2$, with C a uniform constant independent of $z \in \mathbb{B}$ when $|a| \leq 1/2$. Thus $T_{\pm a}$ is the translation by $\mp h$ up to small second order terms, when $|a|$ is small enough.

Step 2: Estimating the invariant symmetric differences of U. The invariant symmetric differences of U in $\mathbb{B}$ are

$$z \in \mathbb{B} \longmapsto U \circ T_a(z) + U \circ T_{-a}(z) - 2U(z) \in \mathbb{R}.$$

We also consider the plurisubharmonic function

$$z \mapsto V_a(z) := \frac{1}{2}(U \circ T_a(z) + U \circ T_{-a}(z)),$$

and try to compare it to U in $\mathbb{B}$. We claim that

$$V_a(z) \leq U(z) + C|a|^2,$$

for some constant $C > 0$ and for all $z, a \in \mathbb{B}$ with $|a| \leq 1/2$. We verify this by showing that V_a belongs to the Perron-Bremermann class $\mathcal{V}(\mathbb{B}, \varphi, f)$, i.e. V_a is a subsolution to the Dirichlet problem.

2.1 Boundary values of V_a. Observe that if $g \in C^{0,1}(\bar{\mathbb{B}})$, then

$$|g \circ T_a(z) - g(z-h)| \leq \|g\|_{C^{0,1}(\bar{\mathbb{B}})} \cdot |T_a(z) - z + h| \leq c_1|a|^2 \|g\|_{C^{0,1}(\bar{\mathbb{B}})}, \tag{4.3}$$

where $c_1 > 0$ is a geometric constant. Using Taylor's expansion we get

$$g \circ T_a(z) = g(z - h + O(|a|^2)) = g(z) + dg(z).h + O(|a|^2)$$

for $g \in C^{1,1}(\bar{\mathbb{B}})$, hence

$$g \circ T_a(z) + g \circ T_{-a}(z) \leq 2g(z) + 2C_2|a|, \tag{4.4}$$

where $C_2 = C_2(g)$ depends on the $C^{1,1}$-norm of g.

Extending φ as a function in $C^{1,1}(\bar{\mathbb{B}})$ and applying (4.4) yields

$$\varphi \circ T_a + \varphi \circ T_{-a} \leq 2\varphi + 2C_2|a|^2, \tag{4.5}$$

where $C_2 = C_2(\varphi)$ depends on the $C^{1,1}$-norm of φ. We infer

$$V_a(z) \leq \varphi + C_2|a|^2, \ \zeta \in \partial\mathbb{B}.$$

2.2. Estimating the Monge-Ampère measure of V_a. We now estimate $\Delta_H V_a$ from below, for $H \in \dot{H}_n^+$ fixed. Observe that

$$\Delta_H(U \circ T_a) \geq (\det T_a')^{2/n} \ (f^{1/n} \circ T_a).$$

where det $T_a'(z) = 1 + (n+1)\langle z, a\rangle + O(|a|^2)$, hence

$$(\det T_a'(z))^{2/n} = 1 + \frac{2(n+1)}{n}\langle z, a\rangle + O(|a|^2).$$

Since $f^{1/n} \in C^{1,1}(\bar{\mathbb{B}})$, it follows from (4.4) that

$$f^{1/n} \circ T_a(z) = f^{1/n}(z - h + o(|a|^2)) = f^{1/n}(z) + \psi_1(z,a) + O(|a|^2).$$

setting $\psi_1(z, a) := df^{1/n}(z).h$. An elementary computation yields

$$|\det T_a'(z)|^{2/n}(f^{1/n} \circ T_a(z) \geq f^{1/n}(z) - \frac{2(n+1)}{n}|\langle z, a\rangle \psi_1(z,a)| - C_3|a|^2,$$

and

$$|\langle z, a\rangle \psi_1(z, a)| \leq C_3|z| \cdot |a|^2 \leq C_3|a|^2,$$

where $C_3 > 0$ only depends on $\|f^{1/n}\|_{C^{1,1}(\overline{\Omega})}$. Therefore

$$|\det T_a'(z)|^{2/n}(f^{1/n} \circ T_a(z)) \geq f^{1/n}(z) - C_3|a|^2,$$

hence $\Delta_H(U \circ T_a) \geq f^{1/n}(z) - C_3|a|^2$ and

$$\Delta_H V_a \geq 2f^{1/n} - 2C_3|a|^2.$$

For $|a| \leq 1/2$ we consider the continuous plurisubharmonic function

$$z \mapsto v_a(z) = V_a(z) + C_3|a|^2(|z|^2 - 2).$$

Observe that $v_a \leq \varphi$ on $\partial\mathbb{B}$ and for every $H \in \dot{H}_n^+$,

$$\Delta_H v_a = \frac{1}{2}\Delta_H V_a + C|a|^2 \Delta_H(|z|^2) \geq f^{1/n} - C_3|a|^2 + C_3|a|^2 \geq f^{1/n}.$$

Thus $v_a \in \mathcal{V}(\Omega, \varphi, f)$, hence $v_a \leq U$. Therefore

$$\frac{1}{2}V_a(z) - C_3|a|^2 \leq \frac{1}{2}V_a(z) + C_3|a|^2(|z|^2 - 2) \leq U(z)$$

for $z \in \mathbb{B}$, hence

$$U \circ T_a(z) + U \circ T_{-a}(z) - 2U(z) \leq 2C_3|a|^2,$$

as claimed.

Step 3: Comparing invariant/usual symmetric differences. We now compare $U \circ T_a(z) + U \circ T_{-a}(z)$ and $U(z-h) + U(z+h)$, where h is defined by (4.2). Fix $K \subset \mathbb{B}$ a compact set and $|h|$ small enough.

Applying (4.3) with $g = U$, $z \in K$ and $|h| < \operatorname{dist}(K, \partial\mathbb{B})$, we get

$$\begin{aligned} U(z-h) &+ U(z+h) - 2U(z) \\ &\leq \; U \circ T_a(z) + U \circ T_{-a}(z) - 2U(z) + 2c_1\|U\|_{C^{0,1}(\bar{\mathbb{B}})}|a|^2 \\ &\leq \; (2c_1\|U\|_{C^{0,1}(\bar{\mathbb{B}})} + 2C_3)|a|^2. \end{aligned}$$

Observe that $a \longmapsto h(a,z) = a - \langle z, a\rangle z$ is a non singular endomorphism of $\mathbb{C}^n$ which depends smoothly on $z \in \mathbb{B}$. The inverse mapping $h \mapsto a(h,z)$ is a linear map with norm less than $\frac{1}{1-|z|^2}$ since

$$|h| \geq |a| - |\langle z, a\rangle||z| \geq |a| - |z|^2|a|| \geq |a|(1 - |z|^2).$$

For $z \in K$ and $|h| \leq \operatorname{dist}(K, \partial\mathbb{B})/2$, we infer

$$U(z+h) + U(z-h) - 2U(z) \leq \frac{C_4}{(1-|z|^2)^2}|h|^2,$$

where $C_4 := (2c_1\|U\|_{C^{0,1}(\bar{\mathbb{B}})} + 2C_3)$.

Consider now a convolution with a regularizing kernel χ_ε, $\varepsilon > 0$ small enough. For $z \in K$ and $|h|$ small we obtain

$$U_\varepsilon(z+h) + U_\varepsilon(z-h) - 2U_\varepsilon(z) \leq \frac{C_4}{(1-(|z|+\varepsilon)^2)^2}|h|^2.$$

The Taylor expansion of order two of U_ϵ yields

$$D^2U_\varepsilon(z).h^2 \leq \frac{C_4}{(1-(|z|+\varepsilon)^2)^2}|h|^2 \leq A|h|^2,$$

for $z \in K, h \in \mathbb{C}^n$, where $A := C_4/\mathrm{dist}(K, \partial\mathbb{B})^2$. Since $U_\varepsilon \in PSH(\mathbb{B}_\varepsilon)$

$$D^2U_\varepsilon(z).h^2 + D^2U_\varepsilon(z).(ih)^2 = 4\sum_{j,k} \frac{\partial^2 U_\varepsilon}{\partial z_j \partial \bar{z}_k}.h_j\bar{h}_k \geq 0.$$

Hence for $z \in K$ and $|h|$ small enough,

$$D^2U_\varepsilon(z).h^2 \geq -D^2U_\varepsilon(z).(ih)^2 \geq -A|h|^2.$$

Therefore for $z \in K$ we have a uniform bound $|D^2U_\varepsilon(z)| \leq A$.

The Alaoglu-Banach theorem insures that there exists $g \in L^\infty(K)$ such that D^2U_ε converges weakly to g in $L^\infty(K)$. Since $D^2U_\varepsilon \to D^2U$ in the sense of distributions, we infer $D^2U = g$ in the sense of distributions. The second order derivatives of U therefore exist almost everywhere and are locally bounded in $\mathbb{B}$ with

$$\|D^2U\|_{L^\infty(K)} \leq A,$$

where $A := C_4/\mathrm{dist}(K, \partial\mathbb{B})^2$ and C_4 depends on the $C^{0,1}$ norm of U, the $C^{1,1}$ norm of φ and $f^{1/n}$. We have thus shown that $U \in C^{1,1}_{loc}(\mathbb{B})$. □

In general U does not belong to $C^{1,1}(\bar{\mathbb{B}})$ as Exercise 4.11 shows.

4.2. Solution to the Dirichlet problem. We now show that the Perron-Bremermann envelope is the solution to the Dirichlet problem:

Theorem 4.14. *Assume $0 \leq f^{1/n} \in C^{1,1}(\bar{\mathbb{B}})$ and $\varphi \in C^{1,1}(\partial\mathbb{B})$. Then $U = U(\mathbb{B}, \varphi, f)$ is the unique solution to the Dirichlet problem*

$$\begin{cases} u \in PSH(\mathbb{B}) \cap C(\overline{\mathbb{B}}) & \\ (dd^c u)^n = f\beta^n & \text{in } \mathbb{B} \\ u = \varphi & \text{in } \partial\mathbb{B}. \end{cases} \tag{4.6}$$

Proof. We already know that $U \in C^{1,1}_{loc}(\mathbb{B}) \cap \mathcal{V}(\mathbb{B}, \varphi, f)$ and has the right boundary values. It remains to show that $(dd^cU)^n = f\beta^n$. Since $U \in C^{1,1}_{loc}(\mathbb{B})$, it suffices to show that for almost every $z \in \mathbb{B}$

$$\det\left(\frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}(z)\right) = f(z).$$

The inequality $\geq$ holds almost everywhere since U is a subsolution.

Suppose by contradiction that there exists a point $z_0 \in \mathbb{B}$ at which U twice differentiable and satisfies

$$\det\left(\frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}(z_0)\right) > f(z_0) + \varepsilon,$$

for some $\varepsilon > 0$. Then for any $H \in \dot{H}_n^+$,

$$\Delta_H U(z_0) > (f(z_0) + 2\epsilon)^{1/n}. \tag{4.7}$$

Using the Taylor expansion of U at order 2 at the point z_0, we get

$$U(z) = U(z_0) + ReP(z - z_0) + L(z - z_0) + o(|z - z_0|^2),$$

where P is a complex polynomial of degree 2 and

$$L(\zeta) = \sum_{j,k} \frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}(z_0)\zeta_j \bar{\zeta_k}$$

is the Levi form of U at z_0.

Since L is positive, for any $0 < s < 1$ close to 1, there exists $\delta, r > 0$ small enough such that $B(z_0, r) \Subset \mathbb{B}$, and for $|z - z_0| = r$,

$$U(z) \geq U(z_0) + ReP(z - z_0) + sL(z - z_0) + \delta,$$

Observe that the function defined by

$$w(z) := U(z_0) + ReP(z - z_0) + sL(z - z_0) + \delta$$

is smooth and plurisubharmonic in $\mathbb{C}^n$.

Then the function

$$(4.8) \qquad v(z) = \begin{cases} U(z), & z \in \mathbb{B} \setminus \mathbb{B}(z_0, r) \\ \max\{U(z), w(z)\}, & z \in \mathbb{B}(z_0, r) \end{cases}$$

is plurisubharmonic in $\mathbb{B}$, continuous near $\partial\mathbb{B}$ with $v = \varphi$ in $\partial\mathbb{B}$.

We claim that $\Delta_H v \geq f^{1/n}$ for all $H \in \dot{H}_n^+$. Indeed if A is the complex hessian matrix associated to U at z_0, we have $\Delta_H w = s\,\mathrm{tr}(HA)$. Lemma 4.8 yields $\mathrm{tr}(HA) \geq (\det A)^{1/n}$, hence by (4.7) for $z \in \mathbb{B}(z_0, r)$,

$$\Delta_H w(z) \geq s(f(z_0) + 2\varepsilon)^{1/n} \geq (f(z_0) + \varepsilon)^{1/n},$$

if $s < 1$ is choosen close enough to 1.

Since $f^{1/n}$ is continuous in $\bar{\mathbb{B}}$, shrinking r if necessary, can assume that $(f(z_0) + \varepsilon)^{1/n} \geq f(z)^{1/n}$ for $z \in \mathbb{B}(z_0, r)$, hence

$$\Delta_H w(z) \geq f(z)^{1/n},$$

pointwise in $\mathbb{B}(z_0, r)$. It follows therefore from (3.3) that $\Delta_H v \geq f^{1/n}$.

We infer $v \in \mathcal{V}(\mathbb{B}, \varphi, f)$ hence $v \leq U$ in $\mathbb{B}$. On the other hand $v(z_0) = U(z_0) + \delta > U(z_0)$, a contradiction. The proof is complete. □

Using an approximation process, we can now solve the Dirichlet problem in the unit ball with continuous data:

COROLLARY 4.15. *Assume $\varphi \in C(\partial\mathbb{B})$ and $0 \leq f \in C(\bar{\mathbb{B}})$. Then $U = U(\mathbb{B}, \varphi, f)$ is the solution to Dirichlet problem (4.6).*

PROOF. Let (f_j) be a sequence of smooth positive functions which decrease to f uniformly on $\bar{\mathbb{B}}$. Fix also φ_j C^∞-smooth functions in $\partial\mathbb{B}$ such that φ_j increases to φ uniformly on $\partial\mathbb{B}$.

The upper envelope $U_j := U(\mathbb{B}, \varphi_j, f_j)$ is the unique plurisubharmonic solution to the Dirichlet problem (4.6) with boundary data φ_j and right hand side f_j. Observe that (U_j) is non decreasing in $\mathbb{B}$.

Fix $\varepsilon > 0$ small and set $\rho(z) := |z|^2 - 1$. Note that for $k \geq j$,

$$(dd^c(U_k + \varepsilon\rho)^n \geq (dd^c U_k)^n + \varepsilon^n \beta^n = (f_k + \varepsilon^n)\beta^n.$$

Since (f_k) decreases fo f uniformly in $\bar{\mathbb{B}}$, we can find $j_0 > 0$ such that $f_j \leq f_k + \varepsilon^n$ in $\bar{\mathbb{B}}$ for $k \geq j \geq j_0$, hence

$$(dd^c(U_k + \varepsilon\rho)^n \geq f_j\beta^n.$$

Since $U_k + \varepsilon(|z|^2 - 1) = f_k \leq f_j$ in $\partial\mathbb{B}$, it follows from the comparison principle that $U_k + \varepsilon\rho \leq U_j$ in $\bar{\mathbb{B}}$. Since $U_j \leq U_k$ we infer for $k \geq j \geq j_0$,

$$\max_{\bar{\mathbb{B}}} |U_k - U_j| \leq \varepsilon.$$

The sequence (U_j) therefore uniformly converges in $\bar{\mathbb{B}}$ to a function $u \in PSH(\mathbb{B}) \cap C^0(\bar{\mathbb{B}})$ such that $u = \varphi$ in $\partial\mathbb{B}$. The uniform convergence insures that $(dd^cU_j)^n$ converge to $(dd^cu)^n$ weakly hence $(dd^cu)^n = f\beta^n$.

The comparison principle guarantees that $u = U(\mathbb{B}, \varphi, f)$ is the unique solution to the Dirichlet problem (4.6). □

5. Strictly pseudo-convex domains

5.1. Continuous densities. We generalize here Corollary 4.15 to the case when $\Omega \Subset \mathbb{C}^n$ is a strictly pseudo-convex domain in $\mathbb{C}^n$.

THEOREM 4.16. *Assume $\varphi \in C(\partial\Omega)$ and $0 \leq f \in C(\bar{\Omega})$. The envelope $U = U(\Omega, \varphi, f)$ is the unique solution to Dirichlet problem.*

PROOF. We already know that $U \in PSH(\Omega) \cap C^0(\bar{\Omega})$ and satisfies $(dd^cU)^n \geq f\beta^n$ weakly. It remains then to check that $(dd^cU)^n \leq f\beta^n$.

We use the classical balayage technique. Let $B \Subset \Omega$ be an arbitrary euclidean ball. By Corollary 4.15, we can solve the Dirichlet problem

$$(dd^cu)^n = f\beta^n \text{ in } B \text{ and } u = U \text{ on } \partial B.$$

The comparison principle insures $U \leq u$ in B. It follows therefore from Proposition 4.10 that

$$z \mapsto v(z) = \begin{cases} u(z) & \text{if } z \in B \\ U(z) & \text{if } z \in \overline{\Omega} \setminus B \end{cases}$$

belongs to the class $\mathcal{V}(\Omega, \varphi, f)$ and $v = U = \varphi$ on $\partial\Omega$. We infer $v \leq U$, hence $u = U$ in B so that $(dd^cU)^n = (dd^cu)^n = f\beta^n$ in B. The equality holds in Ω since B was arbitrary. □

5.2. More general densities. We start by extending Theorem 4.16 to the case when the density is merely bounded:

THEOREM 4.17. *Assume $\varphi \in C(\partial\Omega)$ and $0 \leq f \in L^\infty(\Omega)$. Then the envelope $U(\Omega, \varphi, f)$ is a bounded plurisubharmonic function in Ω which is the unique solution to the Dirichlet problem*

$$(5.1) \qquad \begin{cases} u \in PSH(\Omega) \cap L^\infty(\Omega) & \\ (dd^cu)^n = f\beta^n & \text{in } \Omega \\ \lim_{z \to \zeta} u(z) = \varphi(\zeta) & \text{for } \zeta \in \partial\Omega \end{cases}$$

Proof. Let (f_j) be a sequence of continuous functions in $\bar{\Omega}$ which converge to f in $L^1(\Omega)$ and almost everywhere. Theorem 4.16 yields, for each $j \in \mathbb{N}$, a solution $U_j \in PSH(\Omega) \cap C^0(\bar{\Omega})$ such that $(dd^c U_j)^n = f_j \beta^n$ in Ω and $U_j = \varphi$ in $\partial\Omega$.

Set $V_0 := U(\Omega, \varphi, 0)$ and $V_1 := U(\Omega, \varphi, M)$, where M is a uniform L^∞-bound for the f_j's. The comparison principle yields $V_1 \leq U_j \leq V_0$ Hence the U_j's are uniformly bounded.

Extracting and relabelling we can assume that (U_j) converges in $L^1(\Omega)$ and almost everywhere to a bounded plurisubharmonic function U such that $U = (\limsup_j U_j)^*$ in Ω.

We claim that (U_j) converges to U in capacity. Indeed fix a compact set $K \subset \Omega$ and $\delta, \varepsilon > 0$. There exists an open set $G \subset \Omega$ such that $\mathrm{Cap}_\Omega(G) < \varepsilon$ and all the function U_j, U are continuous in $\Omega \setminus G$, by quasocontinuity. Hartogs lemma yields

$$\limsup_{j \to +\infty} \max_{K \setminus G} (U_j - U) \leq 0,$$

hence $\{U_j - U \geq 2\delta\} \subset G$ for $j > 1$ large enough. We infer

$$\lim_{j \to +\infty} \mathrm{Cap}_\Omega(\{U_j - U \geq 2\delta\}) = 0.$$

On the other hand, Lemma 4.18 below shows that for all $j \in \mathbb{N}$,

$$\begin{aligned} \mathrm{Cap}_\Omega(\{U - U_j \geq 2\delta\}) &\leq \delta^{-n} \int_{\{U - U_j \geq \delta\}} (dd^c U_j)^n \\ &\leq M \delta^{-n-1} \int_\Omega (U - U_j)_+ \beta^n. \end{aligned}$$

The right hand side converges to 0 since $U_j \to U$ in L^1, hence

$$\lim_{j \to +\infty} \mathrm{Cap}_\Omega(\{U - U_j \geq 2\delta\}) = 0.$$

Our claim is proved.

We infer $(dd^c U_j)^n \to (dd^c U)^n$ and $(dd^c U)^n = f \beta^n$ weakly in Ω. Since $V_1 \leq U \leq V_0$, Theorem 4.16 shows that U tends to φ at the boundary of Ω.

The comparison principle insures that $U = U(\Omega, \varphi, f)$ is the unique solution to the Dirichlet problem (5.1). ☐

We need to prove the following lemma which was used in the previous proof.

Lemma 4.18. *Assume u, v are bounded plurisubharmonic functions such that $\{u < v\} \Subset \Omega$. Then for all $s, t > 0$*

$$t^n \, Cap_\Omega(\{u - v \leq -s - t\}) \leq \int_{\{u - v \leq -s\}} (dd^c u)^n$$

Proof. Fix $w \in PSH(\Omega)$ s.t. $-1 \leq w \leq 0$, $s, t > 0$ and note that

$$\{u \leq v - s - t\} \subset \{u \leq v - s + tw\} \subset \{u \leq v - s\} \Subset \Omega.$$

The comparison principle thus yields

$$\begin{aligned} t^n \int_{\{u<v-s-t\}} (dd^c w)^n &\leq \int_{\{u<v-s+tw\}} (dd^c(v-s+tw))^n \\ &\leq \int_{\{u<v-s+tw\}} (dd^c u)^n \\ &\leq \int_{\{u<v-s\}} (dd^c u)^n. \end{aligned}$$

The required estimate follows by taking the sup over all such w's. □

When the density f is merely in $L^1_{loc}(\Omega)$, we can show that the existence of a subsolution implies the existence of a solution:

Corollary 4.19. *Fix $\varphi \in C^0(\partial\Omega)$ and $0 \leq f \in L^1_{loc}(\Omega)$. Assume there exists $v \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $v = \varphi$ in $\partial\Omega$ and $(dd^c v)^n \geq f\beta^n$ in the weak sense. Then the Perron-Bremermann envelope $U(\Omega, \varphi, f)$ is the unique solution to the Dirichlet problem (5.1).*

Proof. Set $f_j := \min\{f, j\}$. Then (f_j) is a sequence of bounded densities which increase to f everywhere and in $L^1_{loc}(\Omega)$.

Theorem 4.17 guarantees the existence of a unique $U_j \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $(dd^c U_j)^n = f_j \beta^n$ in Ω and $U_j = \varphi$ in $\partial\Omega$. Set

$$u_\varphi := U(\Omega, \varphi, 0).$$

The comparison principle yields $v \leq U_{j+1} \leq U_j \leq u_\varphi$, therefore (U_j) is uniformly bounded in Ω.

Since (U_j) is non increasing, it converges to a bounded plurisubharmonic function U in Ω such that $v \leq U \leq u_\varphi$. The continuity of the complex Monge-Ampère operator for decreasing sequences insures $(dd^c U)^n = f\beta^n$. The comparison principle implies that $U = U(\Omega, \varphi, f)$ is the unique solution to the Dirichlet problem (5.1). □

Remark 4.20. *The result above is due to Cegrell and Sadullaev* [**CS92**] *who also gave examples of densities $0 \leq f \in L^1(\Omega)$ for which there is no bounded plurisubharmonic subsolution to the Dirichlet problem (5.1) (see Exercise 4.8).*

When $0 \leq f \in L^p(\Omega)$, $p > 1$, Kolodziej has shown in [**Kol98**] *that the Dirichlet problem (5.1) has a unique continuous solution. The case $p = 2$ was proved earlier by Cegrell and Person* [**CP92**].

The balayage procedure used in the proof of Theorem 4.16 is quite classical in Potential Theory. It can be be generalized as follows.

Corollary 4.21. *Let $B \Subset \Omega$ be a ball and $0 \leq f \in L^1(B)$. Fix $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ such that $(dd^c u)^n \geq f\beta^n$ in B. There exists a unique $\hat{u} \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ such that $\hat{u} = u$ in $\Omega \setminus B$, $u \leq \hat{u}$ in B and $(dd^c \hat{u})^n = f\beta^n$ in B. If $f \in C^0(\bar{B})$ and $u \in C^0(\partial B)$, then $\hat{u} \in C^0(\bar{B})$.*

Proof. Let (φ_j) a decreasing sequence of continuous function converging to u in $\bar{B}$. Applying Theorem 4.17 we find $U_j \in PSH(B) \cap L^\infty(\bar{B})$ such that $U_j = \varphi_j$ in ∂B and $(dd^c U_j)^n = f\beta^n$ in B.

The comparaison principle insures that $u \leq U_j \leq U_{j+1}$ in B. Therefore (U_j) converges to a plurisubharmonic function U in B such that $u \leq U \leq U_j$ and $(dd^c U)^n = f\beta^n$ in B. Moreover

$$U^*(\zeta) := \limsup_{B \ni z \to \zeta} U(z) \leq \limsup_{B \ni z \to \zeta} U_j(z) = \varphi_j(\zeta)$$

for any $\zeta \in \partial B$, hence $U^*(\zeta) \leq u(\zeta)$ in ∂B.

The function $\hat{u}$ defined by $\hat{u} = u$ in $\Omega \setminus B$ and $\hat{u} = U$ in B is therefore plurisubharmonic in Ω and has all the required properties. $\square$

5.3. Stability estimates. We address here the following issue: if f_1 and f_2 (resp. φ_1 and φ_2) are close (in an appropriate sense), does it imply that so are $U(f_1, \varphi_2)$ and $U(f_2, \varphi_2)$?

Proposition 4.22. *Fix $\varphi_1, \varphi_2 \in C^0(\partial\Omega)$ and $f_1, f_2 \in C^0(\overline{\Omega})$. The solutions $U_1 = U(\Omega, \varphi_1, f_1)$, $U_2 = U(\Omega, \varphi_2, f_2)$ satisfy*

$$\|U_1 - U_2\|_{L^\infty(\overline{\Omega})} \leq R^2 \|f_1 - f_2\|^{1/n}_{L^\infty(\overline{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)} \tag{5.2}$$

where $R := diam(\Omega)$. In particular if $\varphi \in C^0(\partial\Omega)$ and $f \in C^0(\overline{\Omega})$, then

$$\|U(\Omega, \varphi, f)\|_{L^\infty(\Omega)} \leq R^2 \|f\|_{L^\infty(\overline{\Omega})} + \|\varphi\|_{L^\infty(\partial\Omega}. \tag{5.3}$$

Proof. For $z_0 \in \Omega$ and $R > 0$ such that $B(z_0, R) \subset \Omega$ we set

$$v_1(z) = \|f_1 - f_2\|^{1/n}_{L^\infty(\overline{\Omega})} (|z - z_0|^2 - R^2) + U_2(z)$$

and

$$v_2(z) = U_1(z) + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}.$$

Observe that $v_1, v_2 \in PSH(\Omega) \cap C(\overline{\Omega})$, $v_1 \leq v_2$ in $\partial\Omega$ and $(dd^c v_1)^n \geq (dd^c v_2)^n$ in Ω. It follows therefore from the comparison principle that $v_1 \leq v_2$ on $\overline{\Omega}$. We infer

$$U_2 - U_1 \leq R^2 \|f_1 - f_2\|^{1/n}_{L^\infty(\overline{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}$$

The inequality (5.2) follows by reversing the roles of U_1 and U_2. $\square$

These stability estimates show that the operator

$$\begin{array}{rcl} U_\Omega : C^0(\partial\Omega) \times L^\infty_+(\Omega) & \longrightarrow & PSH(\Omega) \cap L^\infty(\Omega) \\ (\varphi, f) & \longmapsto & U(\Omega, \varphi, f) \end{array}$$

is continuous for the corresponding topologies. Here $L^\infty_+(\Omega)$ denotes the set of all non-negative measurable and bounded densities in Ω. The reader will check in Exercise 4.13 that this stability property does not hold for arbitrary measures.

Remark 4.23. *Finer stability estimates have been established by Cegrell and Persson* [**CP92**] *when $f \in L^2(\Omega)$ and by Kolodziej* [**Kol02**] *when $f \in L^p(\Omega)$, $p > 1$ (see also* [**GKZ08**]*).*

5.4. More general right hand side. We now allow the right hand side to depend on the unknown function, following Cegrell in [**Ceg84**]. We consider the Dirichlet problem.

$$(5.4)\qquad \begin{cases} u \in PSH(\Omega) \cap L^\infty(\Omega) & \\ (dd^c u)^n = e^u f \beta^n & \text{in } \Omega \\ \lim_{z\to\zeta} u(z) = \varphi(\zeta) & \text{in } \partial\Omega \end{cases}$$

where $\varphi \in C(\partial\Omega)$ and $0 \leq f \in L^\infty(\Omega)$.

The class $\mathcal{W}(\Omega, \varphi, f)$ of subsolutions to this Dirichlet problem is the set of all functions $w \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $w^* \leq \varphi$ in $\partial\Omega$ and $(dd^c w)^n \geq e^w f \beta^n$ in Ω. The corresponding upper envelope is

$$(5.5)\qquad W(\Omega, \varphi, f) = W(z) := \sup\{w(z); w \in \mathcal{W}(\Omega, \varphi, f)\}.$$

Theorem 4.24. *Fix $\varphi \in C(\partial\Omega)$ and $0 \leq f \in L^\infty(\Omega)$. Then $W(\Omega, \varphi, f)$ is the unique solution to the Dirichlet problem (5.4).*

Proof. We can always assume that $\varphi \leq 0$ in $\partial\Omega$. Let ρ be a defining function for Ω and $u_0 := U(\Omega, \varphi, 0)$. The function $v_0 := A\rho + u_0$ is a subsolution to the Dirichlet problem (5.4) if we choose $A > 1$ so large that $A^n (dd^c \rho)^n \geq f \beta^n$ in Ω. Thus $\mathcal{W}(\Omega, \varphi, f)$ is non empty.

We prove that the envelope $W(\Omega, \varphi, f)$ is *the* solution by using Theorem 4.17 and applying the Schauder fixed point theorem. Set

$$\mathcal{C} := \{w \in PSH(\Omega) \cap L^\infty(\Omega); v_0 \leq w \leq u_0\}.$$

Observe that $\mathcal{C}$ is compact and convex in $L^1(\Omega)$. It follows from Theorem 4.17 that for each $w \in \mathcal{C}$ there exists a unique function $v = S(w) \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $v = \varphi$ in $\partial\Omega$ and

$$(dd^c v)^n = e^w f \beta^n \text{ in } \Omega.$$

Observe that $(dd^c v)^n \leq f \beta^n \leq (dd^c v_0)^n$, as $w \leq 0$. Since $v = v_0 = u_0$ in $\partial\Omega$, the comparison principle yields $v_0 \leq v \leq u_0$, hence $S(w) \in \mathcal{C}$.

We claim that the operator $S : \mathcal{C} \longrightarrow \mathcal{C}$ is continuous for the L^1-topology. Indeed assume $(w_j) \in \mathcal{C}^{\mathbb{N}}$ converges to $w \in \mathcal{C}$ in $L^1(\Omega)$ and set $v_j := S(w_j)$. Extracting and relabelling, we can assume that $v_j \to v$ in $\mathcal{C}$ and $w_j \to w$ almost everywhere.

The sequence (v_j) converges to v in capacity. Indeed fix $\delta > 0$. Since $(dd^c v_j)^n = e^{w_j} f \beta^n$ and $w_j \leq 0$, Lemma 4.18 yields for all $j \in \mathbb{N}$,

$$\begin{aligned} \mathrm{Cap}_\Omega(\{v - v_j \geq 2\delta\}) &\leq \delta^{-n} \int_{\{v - v_j \geq \delta\}} (dd^c v_j)^n \\ &\leq \delta^{-(n+1)} \|f\|_{L^\infty(\Omega)} \int_\Omega (v - v_j)_+ \beta^n, \end{aligned}$$

The right hand side converges to 0 for $v_j \to v$ in $L^1(\Omega)$, hence the claim. We infer $(dd^c v_j)^n \to (dd^c v)^n$ weakly in Ω.

On the other hand (w_j) is uniformly bounded in Ω and $e^{w_j}f \to e^w f$ in $L^1(\Omega)$, hence $(dd^c v)^n = e^w f \beta^n$. Since $v_0 \le v \le u_0$ in Ω it follows that $v = \varphi$ in $\partial\Omega$. The comparison principle now yields $v = S(w)$.

It follows that $S(w_j) \to S(w)$ in $L^1(\Omega)$ hence $S : \mathcal{C} \longrightarrow \mathcal{C}$ is continuous. Schauder fixed point theorem insures that S admits a fixed point $w \in \mathcal{C}$, i.e. w is a solution to the Dirichlet problem (5.4).

The uniqueness of solutions to the Dirichlet problem (5.4) is a consequence of the lemma to follow. It guarantees that $w = W(\Omega, \varphi, f)$. □

LEMMA 4.25. *Assume* $0 \le f_1 \le f_2$ *and* $w_1, w_2 \in PSH(\Omega) \cap L^\infty(\Omega)$ *are such that* $(dd^c w_1)^n \le e^{w_1} f_1 \beta$ *and* $(dd^c w_2)^n \ge e^{w_2} f_2$ *in* Ω.
If $w_2 \le w_1$ *on* $\partial\Omega$ *then* $w_2 \le w_1$ *in* Ω.

PROOF. We infer from $f_1 \le f_2$ that

$$\int_{\{w_1<w_2\}} e^{w_2} f_1 \beta^n \le \int_{\{w_1<w_2\}} e^{w_2} f_2 \beta^n = \int_{\{w_1<w_2\}} (dd^c w_2)^n.$$

The comparison principle thus yields

$$\int_{\{w_1<w_2\}} (dd^c w_2)^n \le \int_{\{w_1<w_2\}} (dd^c w_1)^n \le \int_{\{w_1<w_2\}} e^{w_1} f_1.$$

Therefore $\int_{\{w_1<w_2\}} e^{w_2} f_1 \beta^n \le \int_{\{w_1<w_2\}} e^{w_1} f_1 \beta^n$, hence

$$\int_{\{w_1<w_2\}} (e^{w_2} - e^{w_1}) f_1 \beta^n = 0 \;\text{ and }\; \mathbf{1}_{\{w_1<w_2\}} (e^{w_2} - e^{w_1}) = 0,$$

almost everywhere with respect to $\mu_1 := f_1 \beta^n$.

Since $(dd^c w_1)^n \le \mu_1$ we infer $w_2 \le w_1$ almost everywhere for $(dd^c w_1^n)$. The domination principle now yields $w_2 \le w_1$ everywhere. □

5.5. Further results. We mention here without proof a few important results for the sake of completeness.

THEOREM 4.26 (Krylov). *Let* $\Omega \subset \mathbb{C}^n$ *be a smoothly bounded strictly pseudoconvex domain and fix* $\varphi \in \mathcal{C}^{3,1}(\partial\Omega)$. *The unique plurisubharmonic solution* $U = U_{\Omega,\varphi,0}$ *of the homogeneous complex Monge-Ampère equation in* Ω *with boundary values* φ *is* $\mathcal{C}^{1,1}$*-smooth on* $\overline{\Omega}$.

This result (together with many other regularity results) has been obtained by Krylov by probabilistic methods, in a series of articles (see notably [**Kry89**]). We refer the interested reader to the lecture notes by Delarue [**Del12**] for an overwiev of these techniques.

THEOREM 4.27 (Caffarelli-Kohn-Nirenberg-Spruck). *Let* $\Omega \subset \mathbb{C}^n$ *be a smoothly bounded strictly pseudoconvex domain and fix* $\varphi \in \mathcal{C}^\infty(\partial\Omega)$. *If the density* f *is smooth and strictly positive on* $\overline{\Omega}$, *then the unique solution* $U = U_{\Omega,\varphi,f}$ *is smooth up to the boundary.*

This result (together with many others) has been obtained by Caffarelli, Kohn, Nirenberg and Spruck in [**CKNS85**]. We refer the interested reader to the lecture notes by Boucksom [**?**] for an up-to-date presentation.

6. Exercises

EXERCISE 4.1. *Let μ be a probability measure with compact support in $\mathbb{R}^N$.*

1. Show that the function

$$x \in \mathbb{R}^N \mapsto U_\mu(x) := \int_{\mathbb{R}^n} K_N(x-y)d\mu(y) \in \mathbb{R},$$

is subharmonic and satisfies the Poisson equation $\Delta U_\mu = \mu$ in $\mathbb{R}^N$.

2. Let $D \Subset \mathbb{R}^N$ be a domain and u a subharmonic function in a neighborhood of $\overline{D}$. Show that u can be written, in D,

$$u = U_\mu + h_D,$$

where μ is the Riesz measure of u and h_D is harmonic in D.

EXERCISE 4.2. *Let $u \in SH(\mathbb{B}) \cap C^0(\bar{\mathbb{B}})$. Show that for all $x \in \mathbb{B}$,*

$$u(x) = \int_{\mathbb{S}} u(y)P(x,y)d\sigma(y) + \int_{\mathbb{B}} G(x,y)d\mu_u(y),$$

where $\mu_u := \Delta u$ is the the Riesz measure of u.

EXERCISE 4.3. *Let φ be a continous function on $\partial\mathbb{B}$. Show that*

$$P_\varphi(x) := \int_{|y|=1} \varphi(y)P(x,y)d\sigma(y),\ x \in \mathbb{B},$$

defines a harmonic function in the ball $\mathbb{B}$, which is continuous on $\bar{\mathbb{B}}$ and satisfies $P_\varphi(x) = \varphi(x)$ for $x \in \mathbb{S}$.

EXERCISE 4.4. *Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}$-linear isomorphism and $q : \mathbb{C}^n_\zeta \to \mathbb{C}^n_z$ be a C^2-smooth function in a neighborhood of a point z_0. Set $z = T(\zeta)$ and $q_T(\zeta) := q(z) = q(T(\zeta)$.*

1) Check that q_T is C^2-smooth near $\zeta_0 := T^{-1}(z_0)$ and

$$\Delta q_T(\zeta) = \sum_{j=1}^{n} \frac{\partial^2 q_T(\zeta)}{\partial \zeta_j \partial \bar{\zeta}_k} = \mathrm{tr}(T^* Q(z) T),$$

where T^* denotes the complex conjugate transpose of $T := (\frac{\partial z_j}{\partial \zeta_k})$ and $Q(z) := (\frac{\partial^2 q}{\partial z_j \partial \bar{z}_k}(z))$ is the complex hessian of q at z.

2) Fix $H \in \dot{H}_n$. Show that there exists a unitary complex matrix U such that $U^* H U = D$ is a diagonal matrix with positive entries. Set $T = D^{1/2} U$. Check that T is hermitian, $T^* T = H$ and

$$\mathrm{tr}(T^* A T) = \mathrm{tr}(H T^{-1} A T) = \mathrm{tr}(HA)$$

for all hermitian matrix A.

3) Apply 2) with $A(z)$ the complex hessian of q at $z = T(\zeta)$ to get

$$\Delta q_T(\zeta) = \Delta_H q(z).$$

EXERCISE 4.5. *Let μ be a non-negative Borel measure in $\Omega \subset \mathbb{C}^n$ such that there exists $u \in PSH(\Omega) \cap L^\infty(\Omega)$ with $\mu \leq (dd^c u)^n$ in Ω.*

Show that that $PSH(\Omega) \subset L^1_{loc}(\mu)$ and for any compact subsets $K \subset E \subset \Omega$ with $K \subset E^\circ$, there exists $C > 0$ such that

$$\int_K |V| d\mu \leq C \int_E |V| d\lambda,$$

for all $V \in PSH(\Omega)$, where λ is the Lebesgue measure on Ω.

EXERCISE 4.6. *Let u be a plurisubharmonic function in $\Omega \Subset \mathbb{C}^n$.*

1) Assume that there exists constants $A, \delta > 0$ such that

$$u(z+h) + u(z-h) - 2u(z) \leq A\|h\|^2,$$

for all $0 < \|h\| < \delta$ and for all $z \in \Omega$ such that $\mathrm{dist}(z, \partial\Omega) > \delta$.

Show that u is $\mathcal{C}^{1,1}$-smooth and its second derivatives, which exist almost everywhere, satisfy $\|D^2 u\|_{L^\infty(\Omega)} \leq A$.

2) Show that the Monge-Ampère measure $(dd^c u)^n$ is absolutely continuous with respect to the Lebesgue measure dV in Ω, with

$$(dd^c u)^n = c_n \det\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{z_j} \partial \bar{\mathrm{z}}_\mathrm{k}}\right) \mathrm{dV},$$

for some constant $c_n > 0$. Check that this last result actually holds whenever u belongs to the Sobolev space $W^{2,n}_{loc}$.

EXERCISE 4.7. *Let $\mathbb{B}$ denote the unit ball. For $a \in \mathbb{B}$, we set*

$$T_a(z) = \frac{P_a(z) - a + \sqrt{1-|a|^2}(z - P_a(z))}{1 - \langle z, a\rangle} \quad ; \quad P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a$$

where $\langle \cdot, \cdot \rangle$ denote the Hermitian product in $\mathbb{C}^n$.

Check that T_a is a holomorphic automorphism of the unit ball such that $T_a(a) = 0$ and that T_0 is the identity.

EXERCISE 4.8. *Fix $\alpha > 1$ and set*

$$f_\alpha(z) := \frac{1}{|z|^{2n}(1 - \log|z|)^\alpha},$$

1. Check that $f_\alpha \in L^1(\mathbb{B}) \setminus L^p(\mathbb{B})$ for all $p > 1$ and show that there is no $u \in PSH(\mathbb{B}) \cap L^\infty(\mathbb{B})$ such that $(dd^c u)^n \geq f\beta^n$ in $\mathbb{B}$.

2. Show that there exists a unique radial plurisubharmonic function U in $\mathbb{B}$ which is smooth in $\mathbb{B} \setminus \{0\}$ and such that $U(z) = 0$ in $\partial\mathbb{B}$, $(dd^c U)^n = f\beta^n$ in $\mathbb{B}$ and $U(0) = -\infty$.

EXERCISE 4.9. *Give an example of a non-negative Borel measure μ on Ω such that the class $\mathcal{B}(\Omega, \varphi, f)$ is non empty but contains no element u such that $\lim u(z) = \varphi(\zeta)$ for all $\zeta \in \partial\Omega$ (see* [**CS92**]*).*

EXERCISE 4.10. *Let $h : \Omega \longrightarrow \mathbb{R}$ be an upper semi-continuous function. The plurisubharmonic envelope of h is defined, for $z \in \Omega$, by*

$$P_\Omega h(z) := \sup\{u(z); u \in PSH(\Omega); u \leq h \ \ in \ \ \Omega\}.$$

1. Show that $P_\Omega h$ is a plurisubharmonic function in Ω. Observe that $P_\Omega h = h$ is h is plurisubharmonic in Ω.

2. Show that if $h : \Omega \longrightarrow \mathbb{R}$ is continuous then $P_\Omega h$ is continuous in Ω and satisfies

$$\mathbf{1}_{\{P_\Omega h < h\}} (dd^c P_\Omega h)^n = 0.$$

3. Show that if $\Omega = \mathbb{B}$ and h is locally $C^{1,1}$, then $P_\mathbb{B} h$ is locally $C^{1,1}$ in $\mathbb{B}$ and satisfies the Monge-Ampère equation:

$$(dd^c P_\mathbb{B} h)^n = \mathbf{1}_{\{P_\mathbb{B} h = h\}} (dd^c h)^n.$$

EXERCISE 4.11. *Let $\mathbb{B} \subset \mathbb{C}^2$ the unit ball. Fix $\alpha > 0$ and set*

$$\varphi(z, w) := (1 + Rez)^\alpha \ for \ (z, w) \in \partial\mathbb{B}.$$

Observe that the function φ is C^∞-smooth in $\partial\mathbb{B}$, except at the point $(-1, 0)$ and near this point we have $1 + \Re z = O(|w|^2 + (\Im z)^2)$.

1) Assume that $\alpha := 1 + \epsilon$ with $0 < \varepsilon \leq 1$. Show that $\varphi \in C^{2,2\epsilon}(\partial\mathbb{B})$ if $\varepsilon \leq 1/2$ and $\varphi \in C^{3,2\varepsilon-1}(\partial\mathbb{B})$ if $1/2 < \varepsilon \leq 1$.

2) Observe that the function defined by $U(z, w) := (1 + Rez)^{1+\epsilon}$, $(z, w) \in \mathbb{B}$, is plurisubharmonic and continuous in $\bar{\mathbb{B}}$ and that it is the unique solution to the homogeneous Dirichlet problem

$$\begin{cases} u \in PSH(\mathbb{B}) \cap C(\overline{\mathbb{B}}) & \\ (dd^c u)^2 = 0 & in \ \ \mathbb{B} \\ u = \varphi & in \ \ \partial\mathbb{B} \end{cases}$$

3) Check that $U \in C^{1,\epsilon}(\bar{\mathbb{B}}) \cap C^\infty(\mathbb{B})$. When $\varepsilon = 1/2$, verify that $\varphi \in C^{2,1}(\partial\mathbb{B})$ and U is no better than $C^{1,1/2}$.

For $1/2 < \alpha < 1$ show that Theorem 4.12 is optimal: the solution is no better than Lipschitz on $\bar{\mathbb{B}}$ when φ in no better than $C^{1,1}$ on $\partial\mathbb{B}$.

EXERCISE 4.12. *Let $f \in L^p(\mathbb{B})$, with $p > 1$, be a radial non-negative desnity and $\varphi \equiv 0$ in $\partial\mathbb{B}$.*

1) Prove that the solution U of the Dirichlet problem $(dd^c U)^n = f\beta^n$ with $U = 0$ on $\partial\mathbb{B}$ is radial.

2) Show that U is given, for $r := |z| < 1$, by

$$U(r) = -\int_r^1 \frac{2}{t} \left(\int_0^t \rho^{2n-1} f(\rho) d\rho \right)^{1/n} dt,$$

3) Check that $U \in C^{0,2-\frac{2}{p}}(\bar{\mathbb{B}})$ for $1 < p < 2$ hence $U \in C^{0,1}(\bar{\mathbb{B}})$ for $p \geq 2$ (see [**Mon86**] *for more details).*

EXERCISE 4.13. *Let φ_j be a sequence of uniformly bounded plurisubharmonic functions in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ such that $\varphi_j \to \varphi$ in L^1 but $(dd^c\varphi_j)^n$ does not converge to $(dd^c\varphi)^n$. By modifying φ_j near $\partial\mathbb{B}$, construct a sequence of probability measures μ_j in $\mathbb{B}$ and plurisubharmonic functions ψ_j in $\mathbb{B}$ such that*

- *the ψ_j's are uniformly bounded and continuous in $\overline{\mathbb{B}}$;*
- *the ψ_j's are solutions of the Dirichlet problem $Dir(\mathbb{B}, 0, \mu_j)$;*
- *the sequence (μ_j) weakly converges to a probability measure μ;*
- *(ψ_j) does not converge to the solution of $Dir(\mathbb{B}, 0, \mu)$.*

This shows that the stability property obtained in Proposition 4.22 does not hold in general. We refer the interested reader to [**CK94, CK06**] *for more information.*

EXERCISE 4.14. *Let K be a compact subset of $\mathbb{C}^n$. Recall that the polynomial hull of K is*

$$\hat{K} := \{z \in \mathbb{C}^n\,;\, |P(z)| \leq \sup_K |P| \text{ for all polynomials } P\}.$$

Fix $\Omega \subset \mathbb{C}^n$ a bounded pseudoconvex domain, $\phi \in \mathcal{C}^\infty(\partial\Omega)$ and set

$$F := \{(z, w) \in \partial\Omega \times \mathbb{C}\,;\, |w| \leq \exp(-\phi(z))\}.$$

Show that

$$\hat{F} = \{(z, w) \in \partial\Omega \times \mathbb{C}\,;\, |w| \leq \exp(-u(z))\},$$

where $u = U_{\Omega,\phi,0}$ is the unique maximal plurisubharmonic function in Ω with ϕ as boundary values.

Bibliography

[Ale39] Alexandrov, A.D., *Almost everywhere existence of the second order differential of a convex function and some properties of convex functions.* Leningrad. Univ. Ann. (Math. Ser.) **37** (1939), 3–35. (Russian)

[AT84] H. J. Alexander, B. A. Taylor: Comparison of two capacities in $\mathbb{C}^n$. Math. Z. **186** (1984), no. 3, 407–417.

[Bed93] Bedford: Survey of pluri-potential theory. Several complex variables (Stockholm, 1987/1988), 48-97, Math. Notes, 38, Princeton Univ. Press, Princeton, NJ, 1993.

[BK77] E. Bedford, M.Kalka: Foliations and complex Monge-Ampère equations. Comm. Pure Appl. Math. **30** (1977), no. 5, 543-571.

[BT76] Bedford, E. Taylor, B.A. *The Dirichlet problem for a complex Monge-Ampère equation.* Invent. Math. **37** (1976), no. 1, 1–44.

[BT82] E. Bedford, B. A. Taylor: A new capacity for plurisubharmonic functions. Acta Math. **149** (1982), no. 1-2, 1–40.

[BT87] E. Bedford, B. A. Taylor: Fine topology, Šilov boundary, and $(dd^c)^n$. J. Funct. Anal. **72** (1987), no. 2, 225–251.
Preprint arXiv:1307.3008
Invent. Math. **200** (2015), no1, 149-200.

[Blo93] Z. Blocki: Estimates for the complex Monge-Ampère operator, Bulletin of the Polish Academy of Sciences **41** (1993), 151-157.

[Blo96] Z. Blocki: The complex Monge-Ampère operator in hyperconvex domains. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23** (1996), no. 4, 721-747.

[Blo02] Z. Blocki: The complex Monge-Ampère operator in pluripotential theory. Lecture notes (2002).

[Blo04] Z. Blocki: On the definition of the Monge-Ampère operator in $\mathbb{C}^2$. Math. Ann. **328** (2004), no. 3, 415-423.

[Blo06] Z. Blocki: The domain of definition of the complex Monge-Ampère operator. Amer. J. Math. **128** (2006), no. 2, 519-530.
To appear in J. Algebraic Geom.

[Bre59] H.Bremermann: On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Silov boundaries. Trans. Amer. Math. Soc. **91** (1959), 246–276.

[CKNS85] L. Caffarelli, J. J. Kohn, L. Nirenberg, J. Spruck: The Dirichlet problem for nonlinear second-order elliptic equations II. Complex Monge-Ampère, and uniformly elliptic, equations. Comm. Pure Appl. Math. **38** (1985), no. 2, 209–252.

[CC95] L. Caffarelli, X.Cabré: Fully nonlinear elliptic equations. American Mathematical Society Colloquium Publications, **43**. American Mathematical Society, Providence, RI, 1995. vi+104 pp.

[Carlehed99] M.Carlehed: *Potentials in pluripotential theory.* Ann. Fac. Sci. Toulouse Math. (6) **8** (1999), no. 3, 439-469.

[Carleson67] L.Carleson: *Selected problems on exceptional sets.* Van Nostrand Mathematical Studies, No. 13 D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London 1967 v+151 pp.

[Car45] H.Cartan: *Théorie du potentiel newtonien: énergie, capacité, suites de potentiels.* Bull. Soc. Math. France **73** (1945), 74-106.

[Ceg83] U. Cegrell: *Discontinuité de l'opérateur de Monge-Ampère complexe.* C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 21, 869-871.

[Ceg84] U.Cegrell: *On the Dirichlet problem for the complex Monge-Ampère operator.* Math. Z. **185** (1984), no. 2, 247–251.

[Ceg88] U.Cegrell : *Capacities in complex analysis. Aspects of Mathematics,* E14. Friedr. Vieweg & Sohn, Braunschweig, 1988. xii+153 pp.

[Ceg98] U. Cegrell: *Pluricomplex energy.* Acta Math. **180** (1998), no. 2, 187–217.

[Ceg04] U. Cegrell: *The general definition of the complex Monge-Ampère operator.* Ann. Inst. Fourier (Grenoble) **54** (2004), no. 1, 159-179.

[CK94] U.Cegrell, S.Kolodziej: *The Dirichlet problem for the complex Monge-Ampère operator.* Perron classes and rotation-invariant measures. Michigan Math. J. . **41** (1994), no. 3, 563-569.

[CK06] U.Cegrell, S.Kolodziej: *The equation of complex Monge-Ampère type and stability of solutions.* Math. Ann. **334** (2006), no. 4, 713-729.

[CP92] U.Cegrell, L.Persson: *The Dirichlet problem for the complex Monge-Ampère operator: stability in L^2.* Michigan Math. J. **39** (1992), no. 1, 145-151.

[CS92] U.Cegrell, A.Sadullaev: *Approximation of plurisubharmonic functions and the Dirichlet problem for the complex Monge-Ampère operator.* Math. Scand. **71** (1992), no. 1, 62-68.

[CLN69] S.S. Chern, H.Levine, L.Nirenberg: Intrinsic norms on a complex manifold. 1969 Global Analysis (Papers in Honor of K. Kodaira) pp. 119-139 Univ. Tokyo Press, Tokyo.

[Cho53] G. Choquet:*Theory of capacities.* Ann. Inst. Fourier, Grenoble **5** (1953–1954), 131-295 (1955).

[Chir89] E. Chirka: *Complex analytic sets.* Mathematics and its Applications (Soviet Series), 46. Kluwer Academic Publishers Group, Dordrecht, (1989).

[CIL92] Crandall, M., Ishii, H. , Lions, P.L. *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. **27** (1992), 1-67.

[Del12] F.Delarue. *Probabilistic approach to regularity. Complex Monge-Ampère Equations and Geodesics in the Space of Kähler Metrics",* 55-198. Lecture Notes in Math. **2038**, Springer, Heidelberg (2012).

[Dem91] J. P. Demailly: *Potential theory in several complex variables.* Survey available at http://www-fourier.ujf-grenoble.fr/ demailly/books.html.

[Dem13] J. P. Demailly: *Applications of pluripotential theory to algebraic geometry.* Pluripotential theory, 143-263, Lecture Notes in Math., 2075, Springer, Heidelberg, 2013.

[ElM80] H.ElMir: *Fonctions plurisousharmoniques et ensembles polaires.* Séminaire Pierre Lelong-Henri Skoda (Analyse). Années 1978/79 (French), pp. 6176, Lecture Notes in Math. **822**, Springer, Berlin, 1980.

[EGZ11] P. Eyssidieux, V. Guedj, A. Zeriahi: Viscosity solutions to degenerate Complex Monge-Ampère equations. Comm.Pure & Appl.Math **64** (2011), 1059–1094.

[GS80] T.Gamelin, N.Sibony: *Subharmonicity for uniform algebras.* J. Funct. Anal. **35** (1980), no. 1, 64-108.

[Gav77] B. Gaveau: *Méthodes de contrôle optimal en analyse complexe I. Résolution d'équations de Monge-Ampère.* J. Functional Analysis **25** (1977), no. 4, 391–411.

[GT83] Gilbarg, N.Trudinger: *Elliptic partial differential equations of second order.* 2nd edition. Grundlehr. der Math.Wiss. **224** (1983), 513pp.

[GKZ08] V. Guedj, S.Kolodziej, A. Zeriahi: *Hölder continuous solutions to Monge-Ampère equations.* Bull.Lond.Math.Soc **40** (2008), 1070-1080.

[GZ17] V. Guedj, A. Zeriahi: *Degenerate complex Monge-Ampère equations.* Tracts in Mathematics, Vol. 26, 2017.

[Gut01] C.E.Gutierrez: *The Monge-Ampère equation.* Progress in Nonlinear Differential Equations and their Applications, 44. Birkhuser Boston, Inc., Boston, MA, 2001. xii+127 pp.

[HL09] R.Harvey, B.Lawson : *Dirichlet Duality and the non linear Dirichlet problem.* Comm. Pure Appl. Math. **62** (2009), 396-443.

[Hor90] L.Hörmander: *An introduction to complex analysis in several variables.* Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990. xii+254 pp.

[Hor94] L.Hörmander: *Notions of convexity.* Progress in Mathematics, **127**. Birkhäuser Boston, Inc., Boston, MA, 1994. viii+414 pp.

[Jos78] B.Josefson:*On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on* $\mathbb{C}^n$. Ark. Mat. **16** (1978), no. 1, 109-115.

[Kis83] C. Kiselman: *Sur la définition de l'opérateur de Monge-Ampère complexe.* Analyse Complexe (Toulouse, 1983), 139–150, Lecture Notes in Math. **1094**, Springer, Berlin, 1984.

[Kis00] C. Kiselman: *Plurisubharmonic functions and potential theory in several complex variables.* Development of mathematics 1950-2000, 655-714, Birkhäuser, Basel, 2000.

[Klim91] M. Klimek: *Pluripotential theory.* London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1991).

[Kol98] S. Kolodziej: *The complex Monge-Ampère equation.* Acta Math. **180** (1998), no. 1, 69–117.

[Kol02] S. Kolodziej: *Equicontinuity of families of plurisubharmonic functions with bounds on their Monge-Ampère masses.* Math. Z. **240** (2002), no. 4, 835847.

[Kol05] S.Kolodziej: *The complex Monge-Ampère equation and pluripotential theory.* Mem. Amer. Math. Soc. **178** (2005), no. 840, 64 pp.

[Kry89] N. V. Krylov: *Smoothness of the payoff function for a controllable diffusion process in a domain.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 66–96. English translation in Math. USSR-Izv. **34** (1990), no. 1, 65–95.

[Lel57] P. Lelong : *Intégration sur un ensemble analytique complexe.* Bull. Soc. Math. France, **85** (1957), 239–262.

[Lel67] P. Lelong : *Fonctionnelles analytiques et fonctions entières (n variables).* Séminaire de Math. Supérieure, 6eme session, été 1967, Presses Univ. de Montreal 1968.

[Lel69] P. Lelong : *Plurisubharmonic functions and positive differential forms.* Gordon and Breach, New-York, and Dunod, Paris, 1969.

[Lel83] P. Lelong: *Discontinuité et annulation de l'opérateur de Monge-Ampère complexe.* P. Lelong-P. Dolbeault-H. Skoda analysis seminar, 1981/1983, 219-224, Lecture Notes in Math., **1028**, Springer, Berlin, 1983.

[LT83] N. Levenberg, B.A.Taylor: *Comparison of capacities in* $\mathbb{C}^n$. Complex analysis (Toulouse, 1983), 162–172, Lecture Notes in Math. **1094** Springer, Berlin, 1984.

[Mon86] D.Monn: *Regularity of the complex Monge-Ampère equation for radially symmetric functions of the unit ball.* Math. Ann. **275** (1986), no. 3, 501-511.

[PS10b] D. H. Phong, J. Sturm: *The Dirichlet problem for degenerate complex Monge-Ampère equations.* Comm. Anal. Geom. **18** (2010), no1, 145-170.

[Ra95] T.Ransford: *Potential theory in the complex plane.* London Mathematical Society Student Texts, 28. Cambridge University Press, Cambridge, (1995). x+232 pp.

[Rud] W. Rudin: Real and complex analysis. Third edition. McGraw-Hill Book Co., New York, 1987.

[Sad81] A.Sadullaev: *Plurisubharmonic measures and capacities on complex manifolds.* Uspekhi Mat. Nauk **36** (1981), no. 4 (220), 53-105, 247.

[ST97] E. B. Saff, V. Totik: *Logarithmic potentials with exterior fields.* Springer-Verlag, Berlin (1997) (with an appendix by T. Bloom).

[Sic69] J.Siciak: *Separately analytic functions and envelopes of some lower dimensional subsets of* $\mathbb{C}^n$. Ann. Polon. Math. **22** (1969), 145-171.

[Sic81] J.Siciak: *Extremal plurisubharmonic functions in* $\mathbb{C}^n$. Ann. Polon. Math. **39** (1981), 175–211.

[T] G. Tian: Canonical metrics in Kähler geometry. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (2000).

[Wa68] R.Walsh: *Continuity of envelopes of plurisubharmonic functions.* J. Math. Mech. **18** (1968) 143–148.

[Wik04] Wiklund, J. *Matrix inequalities and the complex Monge-Ampère operator.* Ann. Polon. Math. **83** (2004), 211-220.

[X96] Y.Xing: *Continuity of the complex Monge-Ampère operator.* Proc. Amer. Math. Soc. **124** (1996), no. 2, 457-467.

[Za76] V. P.Zaharjuta : *Extremal plurisubharmonic functions, orthogonal polynomials, and the Bernstein-Walsh theorem for functions of several complex variables.* Proceedings of the Sixth Conference on Analytic Functions (Krakow, 1974). Ann. Polon. Math. **33** (1976/77), no. 1-2, 137-148.