Plurisubharmonic singularities

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Abstract

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A model example of a plurisubharmonic (*psh*) function is $u = \log|f|$ for a holomorphic mapping f, and the behaviour of the mapping f near its zero set Z_f corresponds to the behaviour of general psh functions u near the points where $u = -\infty$ (points of their *singularities*).

Considered as distributions, the functions $\log|f|$ serve as potentials for the sets Z_f. In the simplest case when f is a holomorphic function, the transition $f \mapsto Z_f$ can be achieved by applying the Laplace operator to $\log|f|$, which gives a measure supported by Z_f with density equal to the multiplicity of the function along the corresponding component of its zero set. Next step is considering this as the integration current $[Z_f]$ (with the multiplicities taken into account). The approach works in the general situation as well. In 1957, Pierre Lelong proved that the trace measure of any closed positive current has density at every point of its support. The main objects of his study were integration currents over analytic varieties; ten years later, the densities for this case were shown to coincide with the multiplicities of the varieties, and since then they have got the name *Lelong numbers* [Th67]. The notion has turned out to be of great importance. In particular, it provides a powerful link between analytic and geometric objects of modern complex analysis. See Lelong's view of the subject in [Le94], [Le95]; a collection of his relevant papers is presented in [Le98].

Further developments in the field rest mainly on technique of Monge-Ampère operators, the key contribution being made by Demailly, see [Dbook]. Among various applications we mention those to algebraic geometry and number theory.

The lectures gives an introduction to central topics of the theory of singularities of psh functions (psh singularities, for brief), by which we mean *behavior of the functions/currents near their singularity points* rather than the behavior of the singularity sets themselves (although the latter will necessarily come into the picture). We are mainly concerned here with various characteristics of the singularities, such as Lelong numbers and their generalizations (in particular, to those for positive closed currents), local indicators, log canonical thresholds, and relations between them.

Basic notions on psh functions and positive currents are assumed; the reader can consult [B93], [BłLN1], [BłLN2], [GuZe17], [H94], [K00a], [Kl91], [Ko98], [Le69], [LeGr86].

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1 Lelong numbers for psh functions

Here we introduce Lelong numbers for psh functions and describe its elementary properties. Sections 1.1 – 1.4 contain standard facts that can be found, for example, in [BłLN1], [GuZe17], [H94], [Le98], [LeGr86]; for Sections 1.5 and 1.6, see [K94] and [LeR99], respectively.

1.1 Psh functions

Since we will be mostly considering a local situation, we restrict ourselves to functions on domains of $\mathbb{C}^n$. Throughout the exposition, Ω is a bounded domain in $\mathbb{C}^n$, and u is a *plurisubharmonic* (*psh*) function in Ω, i.e., an upper semicontinuous function whose restriction to each complex line L is subharmonic in $\Omega \cap L$. The class of all psh functions in Ω is denoted by $\mathrm{PSH}(\Omega)$. Any psh function is locally integrable, and the topology of $\mathrm{PSH}(\Omega)$ is generated by L^1_{loc}-convergence (equivalently, by the weak convergence on compactly supported, continuous – or smooth – functions) .

Denotation 1.1.1 For any $r > 0$ and $p \in \mathbb{N}$, we set:
$\mathbb{B}_r(x) = \{x \in \mathbb{C}^n : |x| < r\}$, $\mathbb{S}_r(x) = \partial\mathbb{B}_r(x)$, $\mathbb{B}_r := \mathbb{B}_r(0)$, $\mathbb{S}_r := S_r(0)$;
$\tau_p = \pi^p/p!$ is the $2p$-volume of the unit ball in $\mathbb{C}^p$;
$\omega_p = 2\pi^p/(p-1)!$ is the $(2p-1)$-volume of the unit sphere in $\mathbb{C}^p$;
dS is the Lebesgue measure on smooth real hypersurfaces in $\mathbb{C}^n$.

1.2 Definition of Lelong number and elementary properties

Denotation 1.2.1 Given $u \in \mathrm{PSH}(\Omega)$, $x \in \Omega$, and $t < \log \operatorname{dist}(x, \partial\Omega)$, denote

$$\Lambda(u, x, t) = \sup\{u(z) : z \in \mathbb{B}_{e^t}(x)\},$$

which is the same as the maximum of u on $\mathbb{S}_{e^t}$, and

$$\lambda(u, x, t) = \omega_n^{-1} \int_{\mathbb{S}_1} u(x + z\, e^t)\, dS(z).$$

Some standard facts of theory of psh functions:

Proposition 1.2.2 *Let $u \in \mathrm{PSH}(\Omega)$. Then*

(i) with t fixed, the functions $x \mapsto \Lambda(u, x, t)$ and $x \mapsto \lambda(u, x, t)$ are continuous and psh in x;

(ii) with x fixed, the functions $t \mapsto \Lambda(u, x, t)$ and $t \mapsto \lambda(u, x, t)$ are convex and increasing in t;

(iii) $u(x) \le \lambda(u, x, t) \le \Lambda(u, x, t)$.

Since psh functions are locally integrable, it is possible to apply the machinery of differential operators. Let

$$\Delta = 4 \sum_k \frac{\partial^2}{\partial z_k \partial \bar{z}_k}$$

be the Laplace operator, then Δu is a positive measure (which is, up to a constant factor, the *Riesz measure* of u considered as a subharmonic function in $\mathbb{R}^{2n}$). Denote

$$\sigma_u(x, r) = \frac{1}{2\pi} \int_{\mathbb{B}_r(x)} \Delta u.$$

Proposition 1.2.3

$$\frac{\sigma_u(x,r)}{\tau_{n-1}r^{2n-2}} = \frac{\partial\lambda(u,x,\log r)}{\partial\log r}, \tag{1.2.1}$$

$\partial/\partial\log r$ being understood as the left derivative.

Proof. Green's formula. □

Definition 1.2.4 Since $\lambda(u,x,t)$ is convex and increasing, the right-hand side of (1.2.1) is increasing in r and so is its left-hand side, hence there exists the limit

$$\nu(u,x) := \lim_{r\to 0}\frac{\sigma_u(x,r)}{\tau_{n-1}r^{2n-2}}, \tag{1.2.2}$$

the Lelong number of u at x.

In other words, the Lelong number of u is *the* $(2n-2)$*-dimensional density of its Riesz measure at* x. When $n=1$, it is precisely the mass charged by the Riesz measure of u at x.

More elementary representations of $\nu(u,x)$ are in terms of the asymptotic behaviour of u near x – namely, as the slope of the convex functions $\lambda(u,x,t)$ and $\Lambda(u,x,t)$ at $-\infty$. To this end, we need a simple fact (repeatedly used in the future):

Lemma 1.2.5 (slope lemma) *If $g(t)$ is an increasing convex function on an interval $I\subset\mathbb{R}$, then the ratio*

$$\frac{g(t)-g(t_0)}{t-t_0}, \quad t_0\in I,$$

increases in t.

Proof. Direct calculation. □

Theorem 1.2.6 *For any psh function u,*

$$\nu(u,x) = \lim_{t\to-\infty}\frac{\lambda(u,x,t)}{t} = \lim_{t\to-\infty}\frac{\Lambda(u,x,t)}{t}. \tag{1.2.3}$$

Proof. (1.2.1), (1.2.2), slope lemma, and Harnack's inequality. □

Note that $u(x) = -\infty$ does not imply $\nu(u, x) > 0$. This happens if and only if the behavior of $u(z)$ near x is controlled by $\log|z - x|$:

Corollary 1.2.7

$$\nu(u, x) = \liminf_{z \to x} \frac{u(z)}{\log|z - x|} = \sup\{\nu > 0 : u(z) \le \nu \log|z - x| + O(1),\ z \to x\}.$$

Furthermore, for any subdomain $\Omega' \subset\subset \Omega$ containing x there exists a constant C such that

$$u(z) \le \nu(u, x) \log|z - x| + C, \quad z \in \Omega'. \tag{1.2.4}$$

Proof. The first line is just a reformulation of (1.2.3). To prove the second statement, let $x = 0$. By the scaling $z \mapsto tz$, $t > 0$, one can assume $\mathbb{B}_1 \subset \Omega'$. Choose $C > 0$ such that $u(z) < C$ on $\partial\Omega'$. Then for any $\epsilon > 0$ one can find a neighborhood ω of x where $u \le u' := (\nu(u, 0) - \epsilon)\log|z|$; in particular, this is true on $\partial\omega$. Thus we have $u - C \le u'$ on $\partial\Omega''$, where $\Omega'' := \Omega' \setminus \overline{\omega}$. Since the function u' is a *maximal psh function*[1] on the open set Ω'', this implies $u - C \le u'$ on Ω'' and thus on the whole Ω', which in view of the abtrary choice of ϵ gives us (1.2.4). □

The two representations in Theorem 1.2.6 imply nice properties of the Lelong number as a functional on psh singularities. Denote PSH_x the collection of all psh functions (germs) at the point x.

Corollary 1.2.8 *Let u_k be a finite collections of psh functions $u_k \in \mathrm{PSH}_x$. Then*

$$\nu\left(\sum_k u_k, x\right) = \sum_k \nu(u_k, x)$$

and

$$\nu\left(\max_k u_k, x\right) = \min_k \nu(u_k, x).$$

[1]Recall that $u \in \mathrm{PSH}(D)$ is *maximal* on D if for any $D' \subset\subset D$ the condition $v \le u$ on $D \setminus D'$ for $v \in \mathrm{PSH}(D)$ implies $v \le u$ on the whole D.

More difficult results are upper semicontinuity of $\nu(u,x)$ as a function of x and its invariance with respect to the choice of coordinates; we will prove this later as consequences of more general statements for *generalized Lelong numbers of currents with respect to psh weights*.

Even more complicated is the celebrated Siu's analyticity theorem.

Definition 1.2.9 For $u \in \mathrm{PSH}(\Omega)$, denote

$$E_c(u) = \{x \in \Omega : \nu(u,x) \geq c\}, \quad c > 0,$$

the *upperlevel sets* for the Lelong numbers.

Theorem 1.2.10 (Siu) *$E_c(T)$ is an analytic variety in Ω.*

We postpone its proof until we get ready.

1.3 Examples

The following can be easily derived from Theorem 1.2.6.

(a) If $u(z) = \log|z|$, then $\nu(u,0) = 1$.

(b) Let $u(z) = \log|f(z)|$ and $f : \Omega \to \mathbb{C}$ be a holomorphic function, $f(x) = 0$. Then $\nu(u,x) = m$, the *multiplicity (vanishing order) of f at x* (the least degree of a monomial in the Taylor expansion of f near x).

(c) If $f = (f_1, \ldots, f_N) : \Omega \to \mathbb{C}^N$ is a holomorphic mapping, $f(x) = 0$, and $u(z) = \log|f(z)| = \frac{1}{2}\log\sum_k |f_k|^2$, then

$$\nu(u,x) = \min_k m_k,$$

where m_k are the multiplicities of the zeros of the components f_k of the mapping f at x.

1.4 Lelong numbers of slices and pull-backs

Fix $x \in \Omega$. Given $y \in \mathbb{C}^n$, let L be the complex line through x and y, and let u_y be the restriction of u to $\Omega_L = \Omega \cap L$ (the *slice of u on L*):

$$u_y(\zeta) := u(x + \zeta y) \in SH(\Omega_L). \tag{1.4.5}$$

Theorem 1.4.1 *$\nu(u_y,0)\ge\nu(u,x)$ for all $y\in\mathbb{C}^n$, and $\nu(u_y,0)=\nu(u,x)$ for all y outside a pluripolar subset A of $\mathbb{C}^n$.*

Proof. The first statement is evident in view of (1.2.3). To prove the second one, assume $u(x)=-\infty$ and consider the family of functions

$$u_\zeta(y)=-\frac{u(x+\zeta y)}{\log|\zeta|},$$

psh in y and negative on any ball $\mathbb{B}_r$ for $0<|\zeta|<\delta_r<1$. Therefore, the upper semicontinuous regularization $v^*(y)$ of the function

$$v(y)=\limsup_{\zeta\to 0} u_\zeta(y)$$

is a negative psh function on $\mathbb{C}^n$ and thus a constant, $v^*(y)\equiv C$. To find the constant C, note first that $v(y)=-\nu(v_y,0)\le-\nu(u,x)$ for all $y\ne 0$. Furthermore, since

$$\nu(u,x)=\omega_n^{-1}\int_{\mathbb{S}_1}\nu(u_y,0)\,dS(y), \tag{1.4.6}$$

we get $C=-\nu(u,x)$. Finally, as is known, the set $\{y:\ v(y)<v^*(y)\}$ is pluripolar[2] in $\mathbb{C}^n$. □

Let now f be a holomorphic mapping $\Omega'\to\Omega$ with $f(x')=x$, and f^*u be the *pull-back* of a function $u\in PSH(\Omega)$, that is, $(f^*u)(z)=u(f(z))$.

It is easy to see that

$$\nu(f^*u,x')\ge\nu(u,x).$$

A (non-elementary) relation in the opposite direction is given by

Theorem 1.4.2 [F99], [K00b] *If $f(U)$ has non-empty interior for every neighbourhood U of x', then there exists a constant C, independent of u, such that $\nu(f^*u,x')\le C\nu(u,x)$ for any function u plurisubharmonic in a neighbourhood of x. No such bound is possible if $f(U)$ has no interior points for some neighbourhood U of x'.*

[2] A set $E\subset\Omega$ is *prulipolar* if there exists $v\in \mathrm{PSH}(\Omega)$, $v\not\equiv-\infty$, such that $v|_E=-\infty$.

1.5 Directional Lelong numbers

More detailed information on behavior of a psh function near its singularity point can be obtained by comparing it with convex functions in $\mathbb{R}^n$ (rather than with convex functions on $\mathbb{R}$).

Denotation 1.5.1 Given $x \in \Omega$ and vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, one considers the polydisk characteristics

$$\lambda(u, x, a) := (2\pi)^{-n} \int_{[0,2\pi]^n} u(x_k + e^{a_k + i\theta_k})\, d\theta,$$

$$\Lambda(u, x, a) := \sup \{u(z) : z \in T_a(x)\},$$

where

$$T_a(x) = \{z : |z_k - x_k| = e^{a_k},\ 1 \le k \le n\}.$$

Similarly to $\lambda(u, x, t)$ and $\Lambda(u, x, t)$ with $t \in \mathbb{R}$, these functions are convex in a and increasing in each a_k,

$$u(x) \le \lambda(u, x, a) \le \Lambda(u, x, a),$$

and $\lambda(u, x, a)$, $\Lambda(u, x, a) \to u(x)$ as $a_k \to -\infty$, $1 \le k \le n$. This justifies

Definition 1.5.2 [K87], [K94] Given a positive vector $a \in \mathbb{R}^n_+$, there exist the limits

$$\lim_{t \to -\infty} \frac{\lambda(u, x, ta)}{t} = \lim_{t \to -\infty} \frac{\Lambda(u, x, ta)}{t} =: \nu(u, x, a), \quad a \in \mathbb{R}^n_+, \qquad (1.5.7)$$

and the value $\nu(u, x, a)$ is called the *directional* (or *Kiselman's*) *Lelong number in the direction a*.

Proposition 1.5.3 *Let* $\mathbf{1} = (1, \dots, 1)$*, then* $\nu(u, x) = \nu(u, x, \mathbf{1})$.

Proof. $\Lambda(u, x, t) \le \Lambda(u, x, t\mathbf{1}) \le \Lambda(u, x, t + \frac{1}{2}\log n)$. ☐

Example 1.5.4 Let $u(z) = \log |f(z)|$ with $f \in \mathcal{O}_x$. In a neighborhood of x,

$$f(z) = \sum_{J \in \omega_x} c_J (z - x)^J, \quad c_J \ne 0,$$

where $\omega_x \subset \mathbb{Z}_+^n$. Then

$$\nu(u,x,a) = \inf\{\langle a, J\rangle :\ J \in \omega_x\}. \tag{1.5.8}$$

Indeed, denote the right hand side of (1.5.8) by I. Then

$$\begin{aligned}\Lambda(u,x,ta) &= \sup_{\theta\in[0,2\pi]^n} \log\left|\sum_{J\in\omega_x} c_J \exp[\langle ta, J\rangle + i\langle\theta, J\rangle]\right| \\ &= t\,I + \sup_{\theta} \log\left|\sum_{J\in\omega_0} c_J \exp[\langle ta, J\rangle - I + i\langle\theta, J\rangle]\right|\end{aligned}$$

and therefore, $t^{-1}\Lambda(u,0,ta) \to I$ as $t \to -\infty$.

1.6 Local indicators

The notion of local indicator was introduced in [LeR99]. Let $u \in \mathrm{PSH}_x$. We consider the directional Lelong numbers $\nu(u,x,a)$ at the x as functions of $a \in \mathbb{R}_+^n$, and we transform them to functions in the unit polydisk $\mathbb{D}^n$:

Definition 1.6.1 The function

$$\psi_u(s) := -\nu(u,x,-s),\ \ s \in \mathbb{R}_-^n,$$

is non-positive, convex in s and increasing in each s_k, so

$$\Psi_{u,x}(z) := \psi_u(\log|z_1|, \ldots, \log|z_n|)$$

is psh in $\mathbb{D}_*^n := \{z : 0 < |z_k| < 1,\ 1 \le k \le n\}$ and thus extends to a (unique) psh function in the unit polydisk $\mathbb{D}^n$, the *local indicator* of u at x.

When $x = 0$, we write it simply Ψ_u.

It is easily checked that the function ψ_u is *positive homogeneous*,

$$\psi_u(c\,s) = c\,\psi_u(s) \quad \forall c > 0,\ \ s \in \mathbb{R}_-^n. \tag{1.6.9}$$

Proposition 1.6.2 *The indicator Ψ_u is a maximal psh function on $\mathbb{D}_*^n$.*

Proof. Let $y \in \mathbb{D}_*^n$ have coordinates $r_k e^{i\theta_k} \neq 0$. Consider the holomorphic curve $\lambda : \omega \to \Gamma_y \subset \mathbb{C}^n$ on a small neighborhood ω of $1 \in \mathbb{C}$, such that $\lambda_k(\zeta) = r_k^\zeta e^{i\theta_k}$. The function $\lambda^*\Psi_u \in SH(\omega)$ depends only on $\mathrm{Re}\,\zeta$ and satisfies $\lambda^*\Psi_u(c\,\zeta) = c\,\lambda^*\Psi_u(\zeta)$ for all $c > 0$. It is then linear and thus harmonic on ω. This implies, since $\lambda(1) = y$, maximality of Ψ_u on $\mathbb{D}_*^n$. □

Note that $\Psi_{\Psi_u} = \Psi_u$, which means that Ψ_u has the same directional Lelong numbers as the function u.

The following bound is a refinement of $u(z) \le \nu(u,0)\log|z| + C$.

Theorem 1.6.3 *For any function $u \in \mathrm{PSH}_0$,*

$$u \le \Psi_u + C \tag{1.6.10}$$

near the origin. More generally, any $u \in \mathrm{PSH}_x$ satisfies

$$u(z) \le \Psi_{u,x}(z - x) + C$$

near x.

Proof. By the slope lemma, for any $s \in \mathbb{R}_-^n$ and $t < t_0 < 0$, we have

$$\frac{\Lambda(u,0,-ts) - \Lambda(u,0,-t_0 s)}{t - t_0} \ge -\psi_u(s),$$

which implies (1.6.10). □

Examples 1.6.4 1) For $u(z) = \log|z|$, $\Psi_u(z) = \sup_k \log|z_k|$.

2) The functions

$$\varphi_a(z) := \max_k a_k^{-1}\log|z_k|, \quad a_k > 0, \tag{1.6.11}$$

are their own indicators.

3) Let $u = \log|f|$, $f : \Omega \to \mathbb{C}^m$, consider the set

$$\omega_0 = \{J \in \mathbb{Z}_+^n : \sum_j \left|\frac{\partial^J f_j}{\partial z^J}(0)\right| \neq 0\}. \tag{1.6.12}$$

As follows from (1.5.8), $\Psi_u(z) = \sup\{\log|z^J| : J \in \omega_0\}$.

2 Lelong numbers for positive closed currents

Up to this moment, we developed an approach to Lelong numbers based on asymptotic properties of psh functions. More information can be obtained by considering them as densities of the Riesz measures. To do this, the measures should be viewed as the trace measures of the corresponding positive closed currents of degree $(1,1)$. This approach can be extended to currents of higher degrees. One of the motivations for such an extension is as follows. When $u = \log|f|$ and $f : \Omega \to \mathbb{C}$, the Lelong number of u at a point x is just the multiplicity of the zero of f at x, while the multiplicities of holomorphic mappings can be characterized as Lelong numbers of currents of higher degrees.

So we pass to Lelong numbers for positive closed currents, starting with recalling some basic notions of the theory of currents. The subject of Sections 2.1 – 2.4 is treated, e.g., in [BłLN1], [GuZe17], [H94], [Kl91], [Le69], [LeGr86], [Ko98]. Sections 2.5 – 2.8 are taken mainly from [D93] (which is actually Chapter III of [Dbook]). More information on Lelong numbers of the currents will be presented later on.

2.1 Positive closed currents

Here are some denotations and basics concerning positive closed currents.

Denotation 2.1.1 Let $\Omega \subset \mathbb{C}^n$ and $p, q \le n$.
$\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the space of smooth functions compactly supported in Ω.
$\mathcal{D}_{p,q}(\Omega)$ is the space of smooth compactly supported differential forms ϕ of bidegree (p,q) on Ω:

$$\phi = \sum_{|I|=p,|J|=q} \phi_{IJ} dz_I \wedge d\bar{z}_J, \quad \phi_{IJ} \in \mathcal{D}(\Omega),$$

with the topology of C^∞-convergence.

The operator $\partial : \mathcal{D}_{p,q}(\Omega) \to \mathcal{D}_{p+1,q}(\Omega)$ is defined by

$$\partial\phi = \sum_{I,J} \sum_{1\le k\le n} \frac{\partial \phi_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J,$$

and $\bar\partial : \mathcal{D}_{p,q}(\Omega) \to \mathcal{D}_{p,q+1}(\Omega)$ is given by

$$\bar\partial\phi = \sum_{I,J} \sum_{1\le k\le n} \frac{\partial \phi_{IJ}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J.$$

The operators

$$d = \partial + \bar\partial, \quad d^c = \frac{\partial - \bar\partial}{2\pi i}$$

are real, and $dd^c = \frac{i}{\pi}\partial\bar\partial$. (There is no general convention on normalizing the operator d^c, some authors use $d^c = i(\bar\partial - \partial)$; we prefer the above one to avoid extra factors $(2\pi)^p$ in the sequel.)

By Stokes' theorem,

$$\int_D d\phi = -\int_{\partial D} \phi$$

for any domain $D \subset \Omega$ and form ϕ such that $d\phi \in \mathcal{D}_{n,n}(\Omega)$.

Definition 2.1.2 *Currents* of *bidimension* (p,q) ($\equiv$ of *bidegree* $(n-p, n-q)$) are elements of the dual space $\mathcal{D}'_{p,q}(\Omega)$, i.e., continuous linear functionals on $\mathcal{D}_{p,q}(\Omega)$.

Any current $T \in \mathcal{D}'_{p,q}(\Omega)$ has a representation

$$T = \sum_{|I|=n-p, |J|=n-q} T_{IJ} dz^I \wedge d\bar z^J, \quad T_{IJ} \in \mathcal{D}'(\Omega).$$

The action of T on ϕ will be written as $\langle T, \phi\rangle$ or $\int T \wedge \phi$.

The topology on $\mathcal{D}'_{p,q}(\Omega)$ (*the weak topology of currents*):

$$T_j \to T \iff \langle T_j, \phi\rangle \to \langle T, \phi\rangle \quad \forall \phi \in \mathcal{D}_{p,q}(\Omega).$$

Differentiation of currents:

$$\langle \partial T, \phi\rangle := (-1)^{p+q+1}\langle T, \partial\phi\rangle, \quad \langle \bar\partial T, \phi\rangle := (-1)^{p+q+1}\langle T, \bar\partial\phi\rangle.$$

Definition 2.1.3 A current $T \in \mathcal{D}'_{p,p}(\Omega)$ is called *positive* ($T \ge 0$) if $\langle T, \phi\rangle \ge 0$ for every differential form $\phi = i\alpha_1 \wedge \bar\alpha_1 \wedge \ldots \wedge i\alpha_p \wedge \bar\alpha_p$ with $\alpha_k \in \mathcal{D}_{1,0}(\Omega)$.

The coefficients of such a current are Borel measures on Ω. Therefore, the action of a positive current $T = \sum T_{IJ} dz^I \wedge d\bar{z}^J$ can be continuously extended to the space of compactly supported forms ϕ with *continuous* coefficients,

$$|\langle T, \phi \rangle| \leq \|T\|_{\operatorname{supp}\phi} \|\phi\|,$$

where $\|T\|_E = \sum |T_{JK}|_E$, $|T_{JK}|_E$ is the total variation of the measure T_{JK} on E and $\|\phi\| = \sup_{K,L,x} |\phi_{KL}(x)|$.

Denotation 2.1.4

$$\beta := \frac{i}{2} \sum_{1 \leq k \leq n} dz_k \wedge d\bar{z}_k = \frac{\pi}{2} dd^c |z|^2$$

is the *standard Kähler form* on $\mathbb{C}^n$, and

$$\beta_p := \frac{1}{p!} \beta^p$$

is the p-dimensional *volume element*.

For every positive current $T \in \mathcal{D}'_{p,p}(\Omega)$, we have

$$\|T\|_E \leq c_n |T \wedge \beta_p|_E.$$

Definition 2.1.5 A current T is called *closed* if $dT = 0$. When $T \in \mathcal{D}'_{p,p}(\Omega)$, this is equivalent to saying that $\partial T = 0$ or $\bar{\partial} T = 0$.

A useful variant of Stokes' theorem reads as follows: *if $T \in \mathcal{D}'_{p,p}(\Omega)$, then*

$$\int_D d\phi \wedge T = - \int_{\partial D} \phi \wedge T$$

for any domain $D \subset\subset \Omega$ and form ϕ such that $d\phi \in \mathcal{D}_{p,p}(\Omega)$.

Denotation 2.1.6 $\mathcal{D}^+_p(\Omega)$ will denote the cone of all *positive closed currents* from $\mathcal{D}'_{p,p}(\Omega)$.

An important tool in the theory of currents is an extension theorem due to Skoda and El Mir .

Definition 2.1.7 Let E be a closed, complete pluripolar set in Ω (that is, $E = \{z \in \Omega :\ u(z) = -\infty\}$ for some $u \in \mathrm{PSH}(\Omega)$) and let $T \in \mathcal{D}_p^+(\Omega \setminus E)$ be a current whose coefficients T_{IJ} have locally finite mass near E. Then the current $\tilde{T} = \sum \tilde{T}_{IJ} dz^I \wedge d\bar{z}^J$ with $\tilde{T}_{IJ}(A) := T_{IJ}(A \setminus E)$ for all Borel $A \subset \Omega$ is called the *simple*, or *trivial*, *extension* of T.

The simple extension $\tilde{T}$ was first introduced by Lelong when studying integration over analytic varieties, see Example 3 below.

Theorem 2.1.8 [SK82], [EM] *In the above conditions,* $\tilde{T} \in \mathcal{D}_p^+(\Omega)$.

Sometimes $\tilde{T}$ is the unique extension of $T \in \mathcal{D}_p^+(\Omega \setminus E)$: for example, if E is an analytic set of dimension smaller than p (this is a particular case of a more general result of the theory of currents).

2.2 Examples of currents

Standard examples are as follows.

1) *Currents generated by psh functions*:

$$u \in PSH(\Omega) \iff \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) \geq 0 \iff dd^c u \in \mathcal{D}_{n-1}^+(\Omega).$$

Furthermore, if $T \in \mathcal{D}_{n-1}^+(\Omega)$ then for any $x \in \Omega$ there is a neighbourhood U and a function $u_U \in PSH(U)$ such that $T = dd^c u_U$ in U.

2) For M a *complex manifold* of dimension p, the current $[M]$ of integration over M is defined as

$$\langle [M], \phi \rangle = \int_M \phi.$$

Then $[M] \in \mathcal{D}_p^+(\Omega)$ (that it is closed, follows from Stokes' theorem).

3) Integration currents over *analytic varieties*. Let A be an analytic variety, i.e., locally $A = \{z : f_\alpha(z) = 0,\ \alpha \in \mathcal{A}\}$, and let $Reg\, A$ be the set of its regular points (where A is locally a manifold). If A is of pure dimension p, define

$$\langle [A], \phi \rangle := \int_{Reg\, A} \phi.$$

Then $[A] \in \mathcal{D}_p^+(\Omega)$. (Non-trivial part is that $[A]$ is closed; this fundamental result is due to P. Lelong [Le57]. Nowadays it can be seen as a consequence of Theorem 2.1.8.)

4) *Holomorphic chains* $T = \sum \alpha_k[A_k] \in \mathcal{D}_p^+(\Omega)$, where $\alpha_k \in \mathbf{Z}_+$ and A_k are analytic varieties of pure dimension p. When $p = n-1$, the holomorphic chains represent positive, or – in the algebraic geometry language – effective, divisors.

2.3 Monge-Ampère currents

Here is a quick overview of the *complex Monge-Ampère operator*

$$(dd^c u)^n := \underbrace{dd^c u \wedge \ldots \wedge dd^c u}_{n}$$

acting on psh functions u or, more generally,

$$dd^c u_1 \wedge \ldots \wedge dd^c u_p \tag{2.3.1}$$

for psh functions $u_1, \ldots, u_p$, $p \le n$. The wedge product cannot be extended from smooth to arbitrary plurisubharmonic functions. However, $(dd^c u)^p$ can be defined as a positive closed current inductively,

$$(dd^c u)^p = dd^c[u(dd^c u)^{p-1}], \quad p = 2, \ldots, n,$$

for all locally bounded psh functions u (Bedford–Taylor). More generally, for any current $T \in \mathcal{D}_p^+(\Omega)$ and a function $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$, the current uT is well defined, has locally bounded mass, and

$$dd^c u \wedge T := dd^c(uT) \in \mathcal{D}_{p-1}^+.$$

For u smooth, this is a classical result; the general case follows, by Lebesgue's dominated convergence theorem, from approximation of u by smooth u_ϵ.

The complex Monge-Ampère operator gives a characterization of maximal psh functions:

Theorem 2.3.1 *A locally bounded psh function u is maximal on a domain $\omega \subset \mathbb{C}^n$ if and only if $(dd^c u)^n = 0$ in ω.*

The following simple, but important technical result will be repeatedly used in the future.

Lemma 2.3.2 *(Boundary localization principle) Let $\Omega' \subset\subset \Omega$, $T \in \mathcal{D}_1^+(\Omega)$, and let $u, v \in \mathrm{PSH}(\Omega') \cap L^\infty(\Omega')$ be such that $u = v$ near $\partial\Omega'$. Then*

$$\int_{\Omega'} dd^c u \wedge T = \int_{\Omega'} dd^c v \wedge T.$$

Obstacles for the definition of the Monge-Ampère current (2.3.1) arise from the singularity sets of the functions u_k. For the operator be well defined, either the singularity set has to be "small" or the functions must not decrease too rapidly to $-\infty$. Having in mind applications to holomorphic mappings, one needs to make restrictions to the singularity sets themselves (since the decay of $\log|f|$ is unavoidably strongest possible).

Definition 2.3.3 The *l-Hausdorff measure* $\mathcal{H}_l$ is defined as

$$\mathcal{H}_l(E) = \lim_{\epsilon\to 0} \inf \sum_j r_j^l$$

where the *infimum* is taking over all covering of E by balls of radii $r_j < \epsilon$.

Theorem 2.3.4 *(i) Let $T \in \mathcal{D}_p^+(\Omega)$, $u_1, \ldots, u_q \in \mathrm{PSH}(\Omega)$, $q \le p$, and let the unbounded loci L_j of all u_j are either compactly supported in Ω or satisfy*

$$\mathcal{H}_{2(p-m+1)}(L_{j_1} \cap \ldots \cap L_{j_m} \cap \operatorname{supp} T) = 0 \qquad (2.3.2)$$

for all choices of indices $j_1 < \ldots < j_m$, $m = 1, \ldots, q$, where $\mathcal{H}_{2(p-m+1)}$ is the $2(p-m+1)$-dimensional Hausdorff measure. Then the currents

$$u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q$$

and

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T := dd^c(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T)$$

are well defined and have locally finite mass. In particular, $(dd^c u)^n$ is well defined when $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}(\Omega \setminus K)$ if $K \subset\subset \Omega$.

(ii) The Monge-Ampère operators are continuous under monotone limits (and with respect to convergence in capacity).

When the functions u_j have the form $u_j = \log|f_j|$, condition (2.3.2) with $T = 1$ means

$$\dim Z_{j_1} \cap \ldots \cap Z_{j_m} \le n - m, \quad m = 1, 2, \ldots, q, \tag{2.3.3}$$

and for the function $u = \log \sum_{1\le j\le N} |f_j|^2 =: \log|f|^2$, the operator $(dd^c u)^q$ is well defined if

$$\dim Z_f \le n - q, \tag{2.3.4}$$

$Z_f = Z_1 \cap \ldots \cap Z_N$ being the zero set of the mapping $f = (f_1, \ldots, f_N)$.

2.4 Lelong numbers for positive closed currents

Definition 2.4.1 For $T \in \mathcal{D}_p^+(\Omega)$, $\sigma_T := T \wedge \beta_p \in \mathcal{D}_0^+$ is the *trace measure* of T. (If $T = dd^c u$, then σ_T is just the *Riesz measure* σ_u of u.)

Denote $\sigma_T(x, r) = \sigma_T(\mathbb{B}_r(x))$. It can be also represented in the following form.

Proposition 2.4.2

$$\sigma_T(x, r) = \tau_p\, r^{2p} \int_{\mathbb{B}_r(x)} T \wedge (dd^c \log|z - x|)^p\,.$$

Remark. By Theorem 2.3.4, the current $T \wedge (dd^c \log|z - x|)^p$ is well defined.

Proof. Use the boundary localization principle with $u(z) = |z - x|^2$ and $v(z) = \chi_\epsilon(\log|z - x|)$, where $\chi_\epsilon(t)$ equals e^{2t} for $t > \log r - \epsilon$ and is affine otherwise, tangent to e^{2t} at $t = \log r - \epsilon$. □

Definition 2.4.3 The *Lelong number of a current* $T \in \mathcal{D}_p^+(\Omega)$ at $x \in \Omega$ *is*

$$\nu(T, x) = \lim_{r\to 0} \frac{1}{\tau_p r^{2p}} \int_{B_r(x)} T \wedge \beta_p = \lim_{r\to 0} \int_{B_r(x)} T \wedge (dd^c \log|z - x|)^p\,. \tag{2.4.5}$$

So, the Lelong number of a current $T \in \mathcal{D}_p^+(\Omega)$ can be viewed both as the $2p-$dimensional density of its trace measure σ_T and as the mass charged at x by its "projective" trace measure $T \wedge (dd^c \log|\cdot - x|)^p$.

Corollary 2.4.4 *$\sigma_T(x,r) \ge \tau_p r^{2p} \nu(T,x)$.*

Examples 2.4.5 Lelong numbers of the model examples of currents are as follows.

1) If $T = dd^c u$ for a psh function u, then $\nu(T,x) = \nu(u,x)$. This follows from the original definition (1.2.2) of the Lelong numbers of psh functions since $\sigma_u = \sigma_T$. (For the consistency, one should write $\nu(dd^c u, x)$ instead of $\nu(u,x)$, however we prefer to keep the original denotation for the Lelong numbers of functions since it is standard.)

2) For any complex *manifold* M, $\nu([M],x) = 1$ at all points $x \in M$ (and of course $\nu([M],x) = 0$ for x outside M). This will follow from independence of the Lelong numbers of the choice of coordinates, to be proved later.

3) Much more difficult is the corresponding result for *analytic varieties* (Thie's theorem) saying that the Lelong number $\nu([A],x)$ equals the multiplicity m_x of the variety A at x; we will prove it later as well.

Note that the $2p$-dimensional volume of an analytic variety A in a Borel set D is precisely $\sigma_{[A]}(D)$, which gives the following *volume estimation*: *If $K \subset\subset A$ and $r_0 < \operatorname{dist}(K, \partial\Omega)$, then*

$$\tau_p r^{2p} m_x \le \mathrm{Vol}_{2p}(A \cap B_r(x)) \le C(r_0, K, A)\, r^{2p} \quad \forall r < r_0,\ \forall x \in K.$$

4) By linearity, the Lelong number of a holomorphic chain $T = \sum_k \alpha_k [A_k]$ is

$$\nu(T,x) = \sum_k \alpha_k \nu([A_k], x) = \sum_k \alpha_k\, m_{x,k}.$$

Fundamental properties of the Lelong numbers for positive closed currents will be established by using machinery of generalized Lelong numbers due to Demailly.

2.5 Definition of generalized Lelong numbers

An important notion of *generalized Lelong numbers with respect to psh weights* was introduced and studied by Demailly. The idea is to replace the projective trace measure $T \wedge (dd^c \log|\cdot - x|)^p$ by $T \wedge (dd^c \varphi)^p$ with quite general psh functions φ (weights) with singularity at x. Classical and directional Lelong numbers are particular cases of these ones, with specified weight functions. Moreover, the technique of generalized Lelong numbers gives simple and natural proofs of deep results concerning standard Lelong numbers.

Denotation 2.5.1 Given $\varphi \in PSH(\Omega)$ and $r \in \mathbb{R}$,

$$B_r^\varphi = \{z : \varphi(z) < r\}, \quad S_r^\varphi = \{z : \varphi(z) = r\}.$$

Definition 2.5.2 A psh function φ is *semiexhaustive* if $B_R^\varphi \subset\subset \Omega$ for some $R \in \mathbb{R}$. In particular, $\varphi \in L^\infty_{loc}(\Omega \setminus B_R^\varphi)$ and thus $(dd^c\varphi)^k$ is well defined for all $k \le n$. If, in addition, e^φ is continuous, φ is called a *psh weight*.

Definition 2.5.3 Given $T \in \mathcal{D}_p^+(\Omega)$, define

$$\nu(T, \varphi, r) = \int_{B_r^\varphi} T \wedge (dd^c\varphi)^p$$

and

$$\nu(T, \varphi) = \lim_{r \to -\infty} \nu(T, \varphi, r),$$

the *generalized Lelong number*, or *the Lelong-Demailly number*, with respect to the weight φ.

If $T = dd^c u$ for a psh function u, we will occasionally write, as for the classical Lelong numbers, $\nu(u, \varphi)$ instead of $\nu(dd^c u, \varphi)$.

Examples 2.5.4 Previous variants of the Lelong number:

1) if $\varphi(z) = \log|z - x|$, then $B_r^\varphi = \mathbb{B}_{e^r}(x)$, $\nu(T, \varphi, r) = \nu(T, x, e^r)$ and $\nu(T, \varphi) = \nu(T, x)$.

2) the "directional" weights

$$\varphi_{a,x}(z) := \max_k a_k^{-1} \log|z_k - x_k|, \quad a_k > 0, \qquad (2.5.6)$$

generate the directional Lelong numbers with respect to $(a_1, \dots, a_n)$ (to be shown in Section 2.6).

The following useful formula can be derived by means of Stokes' theorem (as was done for the standard Lelong numbers).

Proposition 2.5.5 *For any convex increasing function $\gamma : \mathbb{R} \to \mathbb{R}$,*

$$\nu(T, \gamma \circ \varphi, \gamma(r)) = \gamma'(r)^p \, \nu(T, \varphi, r),$$

γ' being understood as the left derivative. In particular,

$$\nu(T, \varphi, r) = e^{-2pr} \int_{B_r^\varphi} T \wedge \left(\frac{1}{2} dd^c e^{2\varphi}\right)^p.$$

2.6 Lelong–Jensen formula

The classical Lelong number of a psh function is both the density of its associated measure and an asymptotic characteristic of the function itself. A similar relation for the generalized Lelong numbers exists, too.

Definition 2.6.1 Let φ be a psh weight in Ω, $\varphi_r = \max\{\varphi, r\}$. The *swept out Monge-Ampère measure* is

$$\mu_r^\varphi = (dd^c\varphi_r)^n - \mathbf{1}_r(dd^c\varphi)^n,$$

where $\mathbf{1}_r$ is the characteristic function of $\Omega \setminus B_r^\varphi$.

Proposition 2.6.2 *(i)* $\mu_r^\varphi \geq 0$;

(ii) $\operatorname{supp}\mu_r^\varphi \subset S_r^\varphi$ *and* $\mu_r^\varphi(\Omega) = \mu_r^\varphi(S_r^\varphi) = (dd^c\varphi)^n(B_r^\varphi)$;

(iii) if $(dd^c\varphi)^n = 0$ *on* $\Omega \setminus \varphi^{-1}(-\infty)$, *then* $\mu_r^\varphi = (dd^c\varphi_r)^n$;

Example 2.6.3 If $\varphi = \log|z - x|$, then μ_r^φ is the normalized Lebesgue measure on $S_{e^r}(x)$.

Theorem 2.6.4 (Lelong–Jensen–Demailly formula) *Any* $u \in PSH(\Omega)$ *is* μ_r*-integrable, and*

$$\mu_r^\varphi(u) - \int_{B_r^\varphi} u(dd^c\varphi)^n = \int_{-\infty}^r \nu(u,\varphi,t)\,dt.$$

Proof. Fubini's and Stokes' theorems. □

As a consequence, we have

Theorem 2.6.5 *Let* $(dd^c\varphi)^n = 0$ *on* $\Omega\setminus\varphi^{-1}(-\infty)$, *then* $\mu_r^\varphi(u)$ *is an increasing and convex function of* r, *and*

$$\nu(u,\varphi) = \lim_{r\to-\infty}\frac{\mu_r^\varphi(u)}{r}.$$

Proof. For any $r < r_0$,

$$\mu_r^\varphi(u) = \mu_{r_0}^\varphi(u) - \int_r^{r_0} \nu(u,\varphi,t)\,dt.$$

Since $t \mapsto \nu(u,\varphi,t)\,dt$ is positive and increasing, $r \mapsto \mu_r^\varphi(u)$ is increasing and convex. Therefore, there exists the limit

$$\lim_{r\to-\infty}\frac{\mu_r^\varphi(u)}{r} = \lim_{r\to-\infty}\frac{\mu_r^\varphi(u)-\mu_{r_0}^\varphi(u)}{r-r_0} = \lim_{t\to-\infty}\nu(u,\varphi,t) = \nu(u,\varphi).$$

□

Examples 2.6.6 1) When $\varphi(z) = \log|z-x|$, this is the representation of classical Lelong number $\nu(u,x) = \lim_{t\to-\infty}\lambda(u,x,t)/t$.

2) For a directional weight

$$\varphi(z) = \varphi_{a,0}(z) = \max_k a_k^{-1}\log|z_k|, \quad a_k > 0,$$

B_r^φ is the polydisk $\{|z_k| < e^{ra_k}\}$, the measure $\mu_r^\varphi(u)$ is supported by its distinguished boundary and is rotation invariant in each variable. So, $\mu_r^\varphi(u) = c_a\lambda(u,x,ra)$. Later we will compute $c_a = (a_1\dots a_n)^{-1}$, which gives

$$\nu(u,\varphi_{a,0}) = (a_1\dots a_n)^{-1}\nu(u,0,a).$$

2.7 Semicontinuity

The following two 'qualitative' results are useful when studying families of currents/weights.

The first one shows that the generalised Lelong numbers are semicontinuous with respect to variation of the currents.

Theorem 2.7.1 *If currents $T_k \in \mathcal{D}_p^+(\Omega)$ converge to a current T, then*

$$\limsup_{k\to\infty}\nu(T_k,\varphi) \le \nu(T,\varphi).$$

Proof. Boundary localization principle. □

The second result is semicontinuity of the generalized Lelong numbers with respect to variation of psh weights. Here, however, a stronger condition on convergence of the weights is needed.

Theorem 2.7.2 *If psh weights φ_k and φ are such that* $\exp\{\varphi_k\} \to \exp\{\varphi\}$ *uniformly on compact subsets of Ω, then*

$$\limsup_{k\to\infty}\nu(T,\varphi_k) \le \nu(T,\varphi).$$

Proof. The condition on the weights is equivalent to uniform convergence of $\max\{\varphi_k, t\}$ to $\max\{\varphi, t\}$ for every t. This implies convergence of the currents $T \wedge (dd^c \max\{\varphi_k, t\})^p$ to $T \wedge (dd^c \max\{\varphi, t\})^p$. Then – the boundary localization principle. □

As a consequence, we get

Corollary 2.7.3 *Classical Lelong numbers $\nu(T, x)$ are upper semicontinuous functions of x.*

Proof. Choose the weights $\varphi_k = \log|z - x^{(k)}|$ for $x^{(k)} \to x$. □

2.8 Comparison theorems

Now we present two 'quantitative' results on variation of the generalized Lelong numbers with respect to currents and weights.

The first comparison theorem describes dependence on the psh weights. For a weight φ, denote $L(\varphi) = \varphi^{-1}(-\infty)$.

Theorem 2.8.1 *Let $T \in \mathcal{D}_p^+(\Omega)$ and φ and ψ be psh weights such that*

$$\limsup \frac{\psi(z)}{\varphi(z)} = l < \infty \quad \text{as } z \to L(\varphi),\ z \in \operatorname{supp} T,$$

then $\nu(T, \psi) \le l^p\, \nu(T, \varphi)$. Consequently, if there exists $\lim \psi(z)/\varphi(z) = l$, then $\nu(T, \psi) = l^p\, \nu(T, \varphi)$.

Proof. It suffices to establish $\nu(T, \psi) \le \nu(T, \varphi)$, provided $l < 1$. This will be done again by using the boundary localization principle.

For $c > 0$, denote $u_c = \max\{\psi - c, \varphi\}$. If $t < r$, then, for c big enough, $u_c = \varphi$ on $B_r^\varphi \setminus B_t^\varphi$. Therefore, $\nu(T, \varphi, r) = \nu(T, u_c, r) \ge \nu(T, u_c)$.

On the other hand, since $l < 1$, $u_c = \psi - c$ on B_s^φ for $s << t$ and so, $\nu(T, u_c) = \nu(T, \psi - c) = \nu(T, \psi)$. □

The second comparison theorem (which can be proved by similar arguments) indicates dependence of the generalized Lelong numbers on the Monge-Ampère currents.

Theorem 2.8.2 *Let $T \in \mathcal{D}_p^+(\Omega)$ and u_k, $v_k \in PSH(\Omega)$, $1 \le k \le q$, be such that the currents $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ and $dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge T$ are well defined, $v_k = -\infty$ on $\operatorname{supp} T \cap L(\varphi)$, and*

$$\limsup \frac{u_k(z)}{v_k(z)} = l_k < \infty \text{ as } z \to L(\varphi),\ z \in \operatorname{supp} T \setminus v_k^{-1}(-\infty),\ 1 \le k \le q.$$

Then $\nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, \varphi) \le l_1 \ldots l_q\, \nu(dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge T, \varphi)$.

These results make it possible to obtain relatively simple proofs for fundamental facts on the Lelong numbers of psh functions and positive closed currents. For example, the first comparison theorem immediately implies

Corollary 2.8.3 *The Lelong number of a closed positive current is independent of the choice of local coordinates. In particular, $\nu([M], x) = 1$ for any point x on a complex manifold M.*

We can prove now Thie's theorem on the Lelong numbers of the integration currents along analytic varieties as well.

Corollary 2.8.4 *If x is a point of an analytic variety A, then $\nu([A], x)$ equals the multiplicity of the variety at x.*

Proof. Assume x is a singular point of A. Recall that one can choose a neighbourhood U and local coordinates $(z', z'') \in \mathbb{C}^p \times \mathbb{C}^{n-p}$ such that $A \cap U \subset K = \{(z', z'') :\ |z''| \le C|z'|\}$, $C > 0$. In these coordinates, $x = 0$. Let $\mathbb{B}' \subset \mathbb{C}^p$, $\mathbb{B}'' \subset \mathbb{C}^{n-p}$ be such that $\mathbb{B}' \times \mathbb{B}'' \subset U$. The projection $\rho : A \cap (\mathbb{B}' \times \mathbb{B}'') \to \mathbb{B}'$ is proper and finite, so it is a ramified covering of $\mathbb{B}'$. The number m_x of sheets of the covering ρ is the *multiplicity* of A at x.

Set $\varphi(z) = \log |z| = \frac{1}{2} \log(|z'|^2 + |z''|^2)$, $\psi(z) = \log |z'|$. Then $\varphi(z)/\psi(z) \to 1$ as $z \to 0$ inside the cone K (and thus, on A). Therefore,

$$\nu([A], x) = \nu([A], \varphi) = \nu([A], \psi).$$

Furthermore,

$$\begin{aligned} \nu([A], \psi) &= r^{-2\pi} \int_{B^{\psi}_{\log r}} [A] \wedge \left(\frac{1}{2} dd^c e^{2\psi}\right)^p \\ &= r^{-2p} \int_{Reg\, A \cap \{z : |z'| < r\}} \left(\frac{1}{2} dd^c |\rho(z)|^2\right)^p \\ &= m_x\, r^{-2p} \int_{\mathbb{B}'_r} (dd^c |z'|)^p = m_x, \end{aligned}$$

which completes the proof. ☐

The second comparison theorem shows that the residual Monge-Ampère mass $(dd^cu)^n(x)$ is a function of asymptotic behaviour of u near x. This leads to the notion of *psh singularities* at x as equivalence classes of psh functions with respect to their asymptotics.

In particular, the functions $\max_j \log|z_j|^{a_j}$ and $\log\sum_j |z_j|^{a_j}$ (with $a_k > 0$) represent the same singularity at 0. Moreover,

$$(dd^c \max_j \log|z_j|^{a_j})^n = (dd^c \log\sum_j |z_j|^{a_j})^n = a_1 \ldots a_n\, \delta_0 \qquad (2.8.7)$$

(the fact was used in relating directional Lelong numbers $\nu(u,x,a)$ to the Lelong numbers $\nu(u,\varphi_{a,x})$ with respect to the directional weights $\varphi_{a,x}$). This can be proved as follows.

By approximation arguments, it suffices to consider only the case of rational $a_j > 0$. Moreover, due to the homogeneity, we can assume them to be even numbers, $a_j = 2m_j$. Note that both the currents are supported at the origin. Hence the first equality follows from the Theorem 2.8.2. To evaluate the mass of the currents there, we use a representation of the Lelong numbers as densities of measures (Proposition 2.5.5). Denote $\varphi(z) = \frac{1}{2}\log\sum|z_j|^{2m_j}$, then

$$\begin{aligned}\int_{B^{\varphi}_{\log r}} (dd^c\varphi)^n &= r^{-2n}\int_{B^{\varphi}_{\log r}} \left(\frac{1}{2}dd^c e^{\varphi}\right)^n = m_1\ldots m_n\, r^{-2n}\int_{\mathbb{B}_r} (dd^c|w|)^n \\ &= m_1\ldots m_n\, 2^n = a_1\ldots a_n.\end{aligned}$$

Another its application is the following result comparing the Lelong number of a wedge product with the Lelong numbers of the factors.

Corollary 2.8.5 *If $dd^cu_1\wedge\ldots\wedge dd^cu_q$ is well defined, then*

$$\nu(dd^cu_1\wedge\ldots\wedge dd^cu_q\wedge T, x) \ge \nu(u_1,x)\ldots\nu(u_q,x)\,\nu(T,x).$$

In particular,

$$(dd^cu)^n(x) \ge \nu(u,x)^n.$$

3 Relative types and integrability index

Now we pass to other characteristics of psh singularities. The first one, *relative type*, is another, quite elementary, generalization of the notion of Lelong number [R06]. The second one, introduced in [Sk72] as *integrability index*[3], in spite of simplicity of the definition, is more involved; in particular, proofs of its fundamental properties require advanced technique of 'hard analysis'. For Section 3.1, see [R06], and for Section 3.2, see [DK01] and [K94].

3.1 Relative types

The classical Lelong number $\nu(u,x)$ can be defined in two equivalent ways:

$$\nu(u,x) = \liminf_{z\to x} \frac{u(z)}{\log|z-x|} = dd^c u \wedge (dd^c \log|\cdot - x|)^{n-1}(\{x\}).$$

The definition as the residual Monge-Ampère mass has turned out extremely fruitful as it reveals its relations to analytic geometry and allows using machinery of differential operators. By replacing the function $\log|\cdot - x|$ with arbitrary psh weights φ in Demailly's generalized Lelong numbers, the theory has become much more flexible and powerful.

On the other hand, the "elementary" definition of the Lelong number as the lower limit is intimately connected with asymptotic behaviour of psh functions near their singularity points. The bound

$$u(z) \le \nu(u,x)\log|z-x| + O(1)$$

gives the best possible bound on $u(z)$ when $z \to x$ in terms of the 'model' singularity $\log|z-x|$.

This motivates introducing another generalization of Lelong numbers by comparing the asymptotic behavior of the psh function to that of model psh weights φ. In order to get a pointwise bound, one needs to impose certain restrictions on the weights. Namely, this works if the weights φ are maximal psh functions on punctured neighborhoods of x, which is characterized by the equation

$$(dd^c\varphi)^n = 0 \text{ outside } x.$$

Such weights will be called *maximal*.

[3] Its reciprocal is known as *log canonical threshold*, or *complex singularity exponent*.

Definition 3.1.1 Given a *maximal weight* φ, the *type of u relative to φ* is

$$\sigma(u,\varphi) = \liminf_{z\to x} \frac{u(z)}{\varphi(z)}.$$

Then, by the same arguments as for $\log|z-x|$, we get the bound

$$u(z) \le \sigma(u,\varphi)\,\varphi(z) + O(1)$$

as $z \to x$.

Example 3.1.2 A model example of relative types are those with respect to the directional weights

$$\varphi_{a,x} = \max_k a_k^{-1} \log|z_k - x_k|, \quad a_k > 0.$$

In this case,

$$\sigma(u,\varphi_{a,x}) = \nu(u,x,a),$$

the directional Lelong numbers.

As is easy to see, the relative types σ are, as the classic Lelong numbers, upper semicontinuous:

Proposition 3.1.3 *(i) If $u_j \to u$ in L^1_{loc}, then $\sigma(u,\varphi) \ge \limsup \sigma(u_j,\varphi)$.*

(ii) If $e^{\varphi_j} \to e^{\varphi}$ uniformly, then $\sigma(u,\varphi) \ge \limsup \sigma(u,\varphi_j)$.

An analogue of the *first comparison theorem* is the inequality

$$\sigma(u,\varphi) \ge \sigma(u,\psi)\,\sigma(\psi,\varphi),$$

which implies $\sigma(u,\psi) = l\,\sigma(u,\varphi)$, provided $\varphi/\psi \to l$. The *second comparison theorem* is as follows: *if* $\liminf u(z)/v(z) = l$, *then* $\sigma(u,\varphi) \ge l\,\sigma(v,\varphi)$. Note that these are much easier results than those for Demailly's generalized Lelong numbers.

Like the classical Lelong number, the relative type has the property

$$\sigma\left(\max_k u_k\right) = \min_k \sigma(u_k)$$

for finitely many u_k, while the type of a sum of the function need not equal the sum of their types (however it cannot be less than that).

By the first comparison theorem (Theorem 2.8.1) for the currents, the generalized Lelong numbers and relative functions are related by

$$\nu(u,\varphi) \geq m_\varphi\, \sigma(u,\varphi),$$

where $m_\varphi = (dd^c\varphi)^n(x)$. It is not known if any relation in the opposite direction holds for a general maximal weight φ.

It is worth mentioning that the relative types as functionals on psh singularities can be characterized by some of their basic properties:

Theorem 3.1.4 *Let a function $\sigma : \mathrm{PSH}_x \to [0,\infty]$ be such that*

(i) $\sigma(cu) = c\,\sigma(u)$ *for all* $c > 0$*;*

(ii) if $u_1 \leq u_2 + O(1)$ *near* x*, then* $\sigma(u_1) \geq \sigma(u_2)$*;*

(iii) $\sigma(\max_k u_k) = \min_k \sigma(u_k)$*,* $k = 1, 2$*;*

(iv) if $u_j \to u$ *in* L^1_{loc}*, then* $\limsup \sigma(u_j) \leq \sigma(u)$*;*

(v) $\sigma(\log|\cdot - x|) > 0$.

Then there exists a maximal psh weight φ such that $\sigma(u) = \sigma(u,\varphi)$ for every $u \in \mathrm{PSH}_x$. The representation is essentially unique: if two weights φ and ψ represent σ, then $\varphi = \psi + O(1)$ near x.

Proof. Let D be a bounded hyperconvex neighbourhood x. Introduce the function

$$\varphi(z) = \sup\{u(z) :\ u \in \mathcal{M}_\sigma\},$$

where $\mathcal{M}_\sigma = \{u \in \mathrm{PSH}^-(D) :\ \sigma(u) \geq 1\}$, and show that $\sigma(\cdot) = \sigma(\cdot,\varphi)$. ☐

As we will see later on, a variant of Siu's analyticity theorem is valid for the relative types, too.

3.2 Integrability index (and log canonical threshold)

Another way of measuring singularity of u is to study local integrability of $e^{-u/\gamma}$ for $\gamma > 0$ (note that since u may be discontinuous, even integrability of e^{-u} near x with $u(x) > -\infty$ is far from being evident).

Definition 3.2.1 The value

$$I(u,x) = \inf\{\gamma > 0 : e^{-u/\gamma} \in L^2_{loc}(x)\} \tag{3.2.1}$$

is called the *integrability index*, or *Arnold multiplicity*, of u at x. Its reciprocal,

$$lct(u) = (I(u,x))^{-1} = \sup\{c > 0 : e^{-c\,u} \in L^2_{loc}(x)\},$$

is called the *log canonical threshold*, or *complex singularity exponent* at x.

Similarly to the Lelong numbers and their generalizations, stronger singularities have greater integrability indices: $I(u,x) \geq I(v,x)$ if $u \leq v + O(1)$, and $I(c\,u,x) = c\,I(u,x)$. Furthermore, as follows from the Hölder inequality,

$$I(u+v,x) \leq I(u,x) + I(v,x) : \tag{3.2.2}$$

if $a > I(u,x)$, $b > I(v,x)$, then

$$\int e^{\frac{-2(u+v)}{a+b}}\,\beta_n \leq \left(\int e^{-2u/a}\,\beta_n\right)^{1/p} \left(\int e^{-2v/b}\,\beta_n\right)^{1/q}$$

with $p = \frac{a+b}{a}$ and $q = \frac{a+b}{b}$.

In the one-dimensional situation, the integrability index coincides with the point mass of the Riesz measure – i.e., with the Lelong number:

Proposition 3.2.2 *If $n = 1$, then $I(u,x) = \nu(u,x)$.*

Proof. Integral representation of u as the sum of the logarithmic potential of its Riesz measure and a harmonic function. □

Remark 3.2.3 As an immediate consequence, we get that *for* $n = 1$ and any u with $I(u,x) > 0$,

$$e^{-u/I(u,x)} \notin L^2_{loc}(x), \tag{3.2.3}$$

which follows from the inequality $u(z) \leq \nu(u,x)\log|z-x| + O(1)$ and the evident relation $|z|^{-1} \notin L^2_{loc}(0)$. The same property holds in several variables as well, however this is a much more complicated result known as *Opennes Conjecture* of Demailly and Kollár (2001) and proved by Berndtsson in 2013. We will come to this shortly.

Examples 3.2.4 $I(\log|z|, 0) = 1/n$. More generally, if $z = (z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k}$, then $I(\log|z'|, 0) = 1/k$.

An important tool is the following *restriction formula*.

Theorem 3.2.5 (Demailly–Kollar) *If Y is a complex manifold and $x \in Y$, then*

$$I(u|_Y, x) \geq I(u, x).$$

The result follows easily from the Ohsawa-Takegoshi theorem (coming soon, see Section 4.2). Note that, in view of the above examples, the integrability index, unlike the Lelong number, is quite sensitive to the dimension of the singularity set and one cannot expect an equality for generic Y.

Integrability indices are related to the Lelong numbers by *Skoda's inequalities*:

Proposition 3.2.6

$$\frac{1}{n}\,\nu(u, x) \leq I(u, x) \leq \nu(u, x),$$

the extremal situations being realized for $u = \log|z_1|$ and $u = \log|z|$.

Proof. The first inequality follows from the bound $u \leq \nu(u, x)\log|z - x| + O(1)$. The second one results from the restriction formula (Theorem 3.2.5) applied to the restriction of u to a generic complex line l where the equalities $I(u|_l, x) = \nu(u|_l, x) = \nu(u, x)$ hold. □

Remark 3.2.7 A more refined lower bound is due to Kiselman:

$$I(u, x) \geq \sup_{a \in \mathbb{R}^n_+} \frac{\nu(u, x, a)}{\sum_j a_j},$$

which follows from the inequality $u \leq \Psi_{u,x} + O(1)$ and a computation of the integrability index for the indicators $\Psi_{u,x}$.

Like various types of Lelong numbers, the integrability index $I(u, x)$ is an upper semicontinuous function of u:

Theorem 3.2.8 *If $u_j \to u$ in $L^1_{loc}(\Omega)$, then*

$$\limsup_{j\to\infty} I(u_j, x) \le I(u, x),\ \ x \in \Omega.$$

Moreover, if $e^{-u} \in L^2_{loc}(x)$, then $e^{-u_j} \to e^{-u}$ in $L^2_{loc}(x)$.

Applying the second statement to $u_j := u/\gamma_j$ with $\gamma_j \searrow I(u, x)$, we get the openness property (3.2.3).

For $n = 1$, a proof of Theorem 3.2.8 can be deduced again from the the representation of the integrability index as the Riesz mass at x. We postpone the case $n > 1$ until we get a corresponding tool.

An analytic object representing the singularities of u is the *multiplier ideal sheaf* $\mathcal{J}(u)$ consisting of germs of holomorphic functions f such that $|f|e^{-u} \in L^2_{loc}$. It is a coherent analytic sheaf. Moreover, if $U \subset\subset \Omega$ is pseudoconvex, then the restriction of $\mathcal{J}(u)$ to U is generated as an $\mathcal{O}_U$-module by a Hilbert basis $\{\sigma_l\}$ of the Hilbert space $H_u(U)$ of holomorphic functions f on U such that $|f|e^{-u} \in L^2(U)$. We will use this quite soon.

4 Analyticity theorems

Deep results on psh singularities (like the celebrated Siu's analyticity theorem) rest on 'hard analysis', mostly on L^2-extension techniques. We present here two main theorems (due to Hörmander-Bombieri-Skoda and Ohsawa-Takegoshi), without proofs, and show how they imply analyticity theorems on upperlevel sets for the characteristics of psh singularities. We also use the L^2-technique to prove Demailly's approximation theorem and fundamental properties of integrability index.

See [BŁLN2], [D92], [Dbook] (Chapters III and VIII), [Fo], [K94], [R09].

4.1 L^2-extension theorems

Plurisubharmonicity assumes no *a priori* analyticity. Nevertheless, analytic varieties appear from any psh function (moreover, from any positive closed current) with 'strong' singularities.

A bridge between plurisubharmonicity and analyticity are L^2-extension theorems based on the Hörmander type results on the $\bar{\partial}$-problem. In particular, the following two statements have great importance in studying singularities of psh functions.

Theorem 4.1.1 (Hörmander–Bombieri–Skoda) *If u is plurisubharmonic on a bounded pseudoconvex domain Ω and $e^{-u} \in L^2_{loc}(x)$ for some $x \in \Omega$, then there exists a holomorphic function f on Ω such that*

$$\int_\Omega |f|^2 e^{-2u}\,\beta_n < \infty$$

and $f(x) = 1$.

Theorem 4.1.2 (Ohsawa–Takegoshi) *Let Y be a p-dimensional affine subspace of $\mathbb{C}^n$, Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, and $u \in PSH(\Omega)$. Then any function $h \in Hol(Y \cap \Omega)$ with*

$$\int_{Y\cap\Omega} |h|^2 e^{-u}\,\beta_p < \infty$$

can be extended to a function $f \in Hol(\Omega)$ and

$$\int_\Omega |f|^2 e^{-u}\,\beta_n \le A(p, n, \operatorname{diam}\Omega) \int_{Y\cap\Omega} |h|^2 e^{-u}\,\beta_p.$$

As we will see, these (and related) results ensure that the considered characteristics of psh singularities bear a certain analyticity feature. Namely, this is semicontinuity of the characteristics in (analytic) Zariski topology (where closed sets are analytic varieties).

4.2 Integrability index

1. By using the Ohsawa–Takegoshi theorem, we can now prove *the restriction formula for the integrability index*: $I(u|_Y, x) \ge I(u, x)$.

Proof of Theorem 3.2.5. Let $\gamma > I(u|_Y, x)$, then

$$\int_{\mathbb{B}(x,r)\cap Y} e^{-2u/\gamma}\,\beta_p < \infty.$$

Then there exists $F \in \mathcal{O}(\mathbb{B}(x,r))$ such that $F|_Y = 1$ and

$$\int_{\mathbb{B}(x,r)} |F|^2 e^{-2u/\gamma}\,\beta_n < \infty.$$

Since $F(x) = 1$, this means $\gamma > I(u, x)$. □

2. Here we show how the Ohsawa–Takegoshi theorem leads to a proof of *the Openness Conjecture* and *semicontinuity of the integrability index*. We will use arguments from [GuaZh15] and [Hi14].

Sketch of proof of Theorem 3.2.8, $n>1$. In the one-dimensional case, the crucial fact is the uniform bound

$$\int_\Omega e^{-2cu_j}\,\beta_1 \le M,\ j\ge j_0$$

for some $c>1$ and all $j>j_0$, which follows from the upper semicontinuity of the Lelong numbers. In higher dimensions, the argument can be substituted by the Ohsawa–Takegoshi theorem.

We will use induction in the dimension. We assume $u,u_j\in \mathrm{PSH}_-(D)$, $\overline{\mathbb{D}^n}\subset\subset D$. By Fubini's theorem,

$$\int_{\mathbb{D}}\int_{\mathbb{D}^{n-1}} e^{-2u(z',z_n)}\,\beta_{n-1}(z')\,\beta_1(z_n)<\infty.$$

Therefore, for any $\epsilon>0$ there exists $\eta>0$ and a set $E\in\mathbb{D}_\eta\setminus\{0\}$ of positive measure such that

$$\int_{\mathbb{D}^{n-1}} e^{-2u(z',w_n)}\,\beta_{n-1}(z')<\frac{\epsilon^2}{|w_n|^2}$$

for every $w_n\in E$. Then we can choose w_n in such a way that $u_j(\cdot,w_n)$ converge to $u(\cdot,w_n)$. By the induction hypothesis, there exists $c>1$ such that

$$\int_{\mathbb{D}^{n-1}} e^{-2c\,u_j(z',w_n)}\,\beta_{n-1}(z')<\frac{\epsilon^2}{|w_n|^2},\ j\ge j_0,$$

as well. By the Ohsawa–Takegoshi theorem, there exist holomorphic functions f_j in $\mathbb{D}^n$ such that $f_j(z',w_n)\equiv 1$ and

$$\int_{\mathbb{D}^n}|f_j|^2e^{-2c\,u_j}\,\beta_n< A\,\frac{\epsilon^2}{|w_n|^2},\ j\ge j_0.$$

One can then prove that for $\epsilon<1/2$, the inequality $|f_j|\ge 1/4$ holds on a neighborhood ω of 0 and so,

$$\int_\omega e^{-2cu_j}\,\beta_n\le M,\ j\ge j_0.$$

The rest of the proof is exactly as in the one-dimensional case. □

3. Now we pass to an *analyticity theorem for the integrability index*. A basic result is Zariski's semicontinuity of the map $x \mapsto I(u, x)$.

Denotation 4.2.1 $IE_c(u) = \{x : I(u, x) \geq c\}$ is the *upperlevel set for the integrability index* (or, equivalently, the lowerlevel set for the log canonical threshold).

Theorem 4.2.2 *If $u \in \mathrm{PSH}(\Omega)$, then $IE_c(u)$ is an analytic subvariety of Ω for all $c > 0$.*

Proof. Denote
$$N_u = \{x \in \Omega : e^{-u} \notin L^2_{loc}(x)\}.$$
It is an analytic variety in Ω. Indeed, if $x \in N_u$, then $f(x) = 0$ for any function f from

$$\mathcal{H}_u = \{f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 e^{-2u} \beta_n < \infty\},$$

so $N_u \subset \cap\{f^{-1}(0) : f \in \mathcal{H}_u\}$. The reverse inclusion follows from Hörmander's Theorem 4.1.1.

Now, since
$$IE_c(u) = \bigcap_{a<c} N_{u/a},$$
this completes the proof. □

4.3 Demailly's approximation theorem

In order to give a short proof of Siu's analyticity theorem for Lelong numbers, we present here an approximation technique due to Demailly, an extremely powerful tool for investigating psh singularities.

Definition 4.3.1 Given $u \in PSH(\Omega)$, take the *multiplier ideals*

$$\mathcal{J}(mu) = \{f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 e^{-2mu} \beta_n < \infty\}, \quad m = 1, 2, \ldots,$$

considered as weighted Hilbert spaces $\mathcal{H}_m = \mathcal{H}_{m,u}(\Omega)$ with the inner products

$$(f,g)_m = \int_\Omega f\,\bar g\, e^{-2mu}\beta_n.$$

Let $\{\sigma_l^{(m)}\}_l$ be an orthonormal basis of $\mathcal{H}_m$, then $K_m(z) := \sum_l |\sigma_l^{(m)}(z)|^2$ is the Bergman kernel for $\mathcal{H}_m$, the series being converging uniformly on compact subset of Ω. Denote

$$u_m = \mathcal{D}_m u = \frac{1}{2m}\log K_m \in \mathrm{PSH}(\Omega).$$

Note that

$$u_m(z) = \frac{1}{m}\sup\{\log|f(z)| : \, ||f||_m < 1\} \tag{4.3.1}$$

because $K_m(z)^{1/2}$ is the norm of the evaluation functional $f \mapsto f(z)$ on $\mathcal{H}_m$.

Theorem 4.3.2 [D92]

(i) There are constants $C_1, C_2 > 0$ such that for any $z \in \Omega$ and every $r < \operatorname{dist}(z, \partial\Omega)$,

$$u(z) - \frac{C_1}{m} \le u_m(z) \le \sup_{\zeta\in B_r(z)} u(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}. \tag{4.3.2}$$

In particular, $u_m \to u$ pointwise and in $L^1_{loc}(\Omega)$.

(ii)

$$\nu(u,x) - \frac{n}{m} \le \nu(u_m, x) \le \nu(u,x) \quad \forall x \in \Omega.$$

(iii)

$$I(u,x) - \frac{1}{n} \le I(u_m, x) \le I(u,x).$$

Proof. For $f \in \mathcal{H}_m$,

$$|f(z)|^2 \le \frac{1}{\tau_n r^{2n}}\int_{\mathbb{B}_r(z)} |f|^2\,\beta_n \le \frac{||f||_m^2}{\tau_n r^{2n}}\, e^{2m\sup\{u(\zeta):\,\zeta\in\mathbb{B}_r(z)\}},$$

which, by (4.3.1), gives us the second inequality in (4.3.2).

The proof of the first inequality uses the Ohsawa–Takegoshi theorem (more exactly, its particular case of a one-point set Y). Namely, for any $z \in \Omega$ and $a \in \mathbb{C}$ there exists a function $f \in \mathcal{O}(\Omega)$ such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\,u} \beta_n \le A\,|a|^2 e^{-2m\,u(z)}.$$

Choosing a such that the RHS here equals 1, we get $||f_m||_m \le 1$ and so,

$$u_m(z) \ge \frac{1}{m} \log|f(z)| = \frac{\log|a|}{m} = u(z) - \frac{\log C}{m}.$$

Assertions (ii) and (iii) follow from (i). □

Remark 4.3.3 It is worth mentioning that the functions u_m from Theorem 4.3.2 control not only classical Lelong numbers of u, but also all its directional ones:

$$\nu(u,x,a) - m^{-1} \sum_j a_j \le \nu(u_m, x, a) \le \nu(u,x,a) \quad \forall x \in \Omega,\ \forall a \in \mathbb{R}^n_+.$$

Moreover, a similar fact is true for the relative types $\sigma(u_m, \varphi)$ with respect to the weights φ that are exponentially Hölder continuous:

$$e^{\varphi(x)} - e^{\varphi(y)} \le C\,|x-y|^\alpha;$$

see [R01], [BoFJ08], [R13].

4.4 Upperlevel sets for Lelong numbers

Given $T \in \mathcal{D}_p^+(\Omega)$, let

$$E_c(T) := \{x \in \Omega : \nu(T,x) \ge c\},\ \ c > 0,$$

be the *upperlevel sets for the Lelong numbers of T*.

Since the function $x \mapsto \nu(T,x)$ is lower semicontinuous (in the usual topology), every set $E_c(T)$ is closed.

Proposition 4.4.1 *$E_c(T)$ has locally finite $2p$-Hausdorff measure.*

Proof. Use the bound for the trace measure σ_T of the current T,

$$\sigma_T(\mathbb{B}_\epsilon(x)) \geq \epsilon^{2p}\nu(T,x),$$

to deduce

$$\mathcal{H}_p(K_c) \leq C \liminf_{\epsilon\to 0} \sigma_T(K_c + \mathbb{B}_\epsilon) < \infty$$

for the sets $K_c = E_c(T) \cap K$, $K \subset\subset \Omega$. □

Next step is a reduction to psh functions:

Theorem 4.4.2 *Given a current $T \in \mathcal{D}_p^+(\Omega)$, there exists $u \in PSH(\Omega)$ such that $\nu(T,x) = \nu(u,x)$ for every x.*

Sketch of a proof. Consider the 'canonical potential'

$$U_j(z) = -\omega_p^{-1} \int |z-\zeta|^{-2p}\eta_j(\zeta)\, d\sigma_T(\zeta)$$

with η_j a non-negative, smooth function supported in Ω, and equal to 1 on a neighbourhood of $\overline{\Omega}_j \subset\subset \Omega$. The potential is subharmonic in $\mathbb{R}^{2n}$, and its Riesz measure

$$\sigma_j(x,r) = \frac{1}{2\pi}\int_{B_r(x)} \Delta U_j = [1+o(1)]\tau_{n-1}r^{2n-2}\nu(T,x) + o(r^{2n-2})$$

as $r \to 0$, so

$$\lim_{r\to 0} \frac{\sigma_j(x,r)}{\tau_{n-1}r^{2n-2}} = \nu(T,x) \quad \forall x \in \Omega_j.$$

One can show that $dd^cU_j \geq -N_j\, dd^c|z|^2$, so $u_j(z) := U_j(z) + N_j\,|z|^2 + M_j$ is psh and $\nu(T,x) = \nu(u_j,x)$ for all $x \in \Omega_j$. Exhausting Ω by Ω_j we get the desired function u. □

The main result is as follows.

Theorem 4.4.3 (Siu's analyticity theorem) *If $T \in \mathcal{D}_p^+(\Omega)$ and $c > 0$, then $E_c(T)$ is an analytic variety of dimension $\leq p$.*

The original Siu's proof (1974) (developing results of Bombieri and Skoda) takes about 100 pages. A considerable simplification was made by Lelong (1977) who reduced the problem to that for psh function. In 1979, Kiselman applied the attenuating singularities technique to get a simpler proof for the classical Lelong numbers and, in 1986, for the directional numbers. His ideas were used by Demailly to prove the theorem for the generalized Lelong numbers (1987). Here we give the shortest known proof for the classical Lelong numbers due to Demailly [D92].

Proof. It remains to show that for any psh function u, the set

$$E_c(u) := \{x \in \Omega : \nu(u,x) \geq c\}$$

is analytic. Let u_m be Demailly's approximations of u by functions with analytic singularities. Then $\nu(u,x) \geq c$ implies

$$\nu(u_m, x) \geq c - \frac{n}{m}.$$

On the other hand, if $\nu(u,x) < c$, then $\nu(u_m,x) < c$ for all m and hence

$$\nu(u_m, x) < c - \frac{n}{m_0}$$

for some m_0. Therefore,

$$E_c(u) = \bigcap_{m \geq m_0} E_{c-n/m}(u_m).$$

We recall that

$$u_m = \frac{1}{2m} \log \sum_l |\sigma_l^{(m)}(z)|^2$$

with analytic functions $\sigma_l^{(m)}$. Since

$$x \in E_{c-n/m}(u_m) \iff \frac{\partial^\alpha}{\partial z^\alpha} \sigma_l^{(m)}(x) = 0 \quad \forall \alpha : |\alpha| < cm - n,$$

each set $E_{c-n/m}(u_m)$ is analytic, and so is $E_c(u)$. □

4.5 Analyticity theorem for Lelong–Demailly numbers

Let X be a Stein manifold (for example, a pseudoconvex domain in $\mathbb{C}^m$), and let φ be a continuous semiexhaustive psh function on $\Omega \times X$ (that is, $\{(z,x) : \varphi(z,x) < C\} \subset\subset \Omega \times X$ for some $C \in \mathbb{R}$). The function $\varphi_x(z) := \varphi(z,x)$ is a psh weight on Ω. Denote

$$E_c = E_c(T, \varphi) = \{x \in X : \nu(T, \varphi_x) \geq c\}.$$

Theorem 4.5.1 [D87], [D93], [Dbook] *If $\exp \varphi \in C(\Omega \times X)$ and is Hölder with respect to x, then E_c is analytic in X.*

Scheme of proof.

1) Construction of a family of psh potentials $u_a(x)$, $a \geq 0$, whose behaviour is determined by $\nu(T, \varphi_x)$. This refined version of what was done for the classical Lelong numbers is the *attenuating singularities technique* due to Kiselman.

2) Let $N_{a,b} = \{x \in X : \exp\{-u_a/b\} \notin L^2_{loc}(x)\}$, then $E_c = \bigcap N_{a,b}$, the intersection taken over all $a < c$ and $b < (c-a)\gamma/m$, where γ is the Hölder exponent of φ.

3) By the analyticity theorem for the integrability indices, each set $N_{a,b}$ is analytic.

4.6 Analyticity for relative types

A similar approach can be applied to the superlevel sets for the types relative to Hölder continuous maximal weights.

We consider a continuous, semiexhaustive function $\varphi \in \mathrm{PSH}(\Omega\times\Omega)$, such that the set $\{z : \varphi(z,x) = -\infty\}$ is finite for every $x \in \Omega$, $(dd^c\varphi)^n = 0$ on $\{(x,y) : \varphi(z,x) > -\infty\}$, and exponentially Hölder continuous in x.

It then follows that $\varphi_x(z) := \varphi(z,x)$ is a weight satisfying $(dd^c\varphi_x)^n = 0$ outside $\varphi_x^{-1}(-\infty)$. It can be shown that $\Lambda(u, \varphi_x, r) = \sup\{u(z) : \varphi_x(z) < r\}$ is psh [D85, Thm. 6.11].

By the scaling $\varphi \mapsto c\,\varphi$, $c > 0$, it suffices to consider the sets

$$S_1(u, \varphi, \Omega) = \{x \in \Omega : u(z) \leq \varphi(z,x) + O(1) \text{ as } z \to x\}.$$

Theorem 4.6.1 [R09] *Let $\varphi \in \mathrm{PSH}(\Omega \times \Omega)$ satisfy the above conditions. Then for any $u \in PSH(\Omega)$, the set $S_1(u,\varphi,\Omega)$ is analytic.*

This can be applied to questions on propagation of (sub)analytic singularities of the following kind.

Corollary 4.6.2 [R09] *Let the zero sets A_j of functions $f_1, \ldots, f_q \in \mathcal{O}(\Omega)$, $q < n$, form a complete intersection, i.e.,* codim $Z = q$, *where $Z = \cap_j A_j$. If a function $u \in \mathrm{PSH}(\Omega)$ satisfies*

$$u \leq \log|f| + O(1) \qquad (4.6.3)$$

on an open set ω intersecting all the irreducible components of Z, then it satisfies (4.6.3) on the whole Ω.

Proof. It suffices to consider the situation when $u < 0$ and there exist functions $f_{q+1}, \dots, f_n \in \mathcal{O}(\Omega)$ such that for every $x \in \Omega$ the set $\{z : f_j(z) = f_j(x), \ 1 \le j \le n\}$ is finite. Denote $f' = (f_1, \dots, f_q)$ and $f'' = (f_{q+1}, \dots, f_n)$, so the bound (4.6.3) rewrites as $u \le \log |f'| + O(1)$.

For $m > 0$, take

$$\varphi_m(z, x) = \max\{\log(|f'(x) - f'(z)|, m \log |f''(x) - f''(z)|\}.$$

It satisfies the conditions of Theorem 4.6.1, so $S_1(u, \varphi_m, \Omega)$ is an analytic variety in Ω such that

$$S_1(u, \varphi_m, \Omega) \cap \omega \supset S(\log |f'|, \varphi_m, \Omega) \cap \omega.$$

Therefore, it contains all irreducible components of $S_1(\log |f'|, \varphi_m, \Omega)$, that is, the set Z.

Given $a \in Z$, we can assume $D = \{x \in \Omega : \ \varphi_1(x, a) < 0\} \subset\subset \Omega$. Then $u \le \varphi_m(x, a)$ on D because the latter is the largest negative psh function v on D such that $\sigma(v, \varphi_m) \ge 1$. Taking $m \to \infty$, we get $u \le \log |f'|$ in D. $\square$

4.7 Siu's decomposition formula

Importance of Siu's fundamental result can be illustrated by structural formulas for closed positive currents.

Definition 4.7.1 Let A be an irreducible analytic variety in Ω, $\dim A = p$. The *generic Lelong number of $T \in \mathcal{D}_p^+(\Omega)$ along A* is

$$\nu(T, A) := \inf \{\nu(T, x) : x \in A\}.$$

By Siu's semicontinuity theorem, $\nu(T, A) = \nu(T, x)$ for all $x \in A$ outside some its proper analytic subvariety A'.

Denote by χ_A the characteristic function of the set A: $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

Proposition 4.7.2 $\chi_A T = \nu(T, A)\, [A]$.

Proof. Note that $\chi_A T \in \mathcal{D}_p^+(\Omega)$. Indeed, $\chi_A T = T - \chi_{\Omega \setminus A} T \ge 0$ and $\chi_{\Omega\setminus A} T$ is close in view of Theorem 2.1.8 (it is the simple extension of T along A). Since $\chi_A T$ is of bidimension (p, p) and supported by the $p-$dimensional variety

A, it has the form $\chi_A T = \lambda[A]$ for a positive locally integrable function λ on $Reg\, A$ (general fact of theory of currents). Since it is closed, λ is a nonnegative constant. Finally, $\lambda = \nu(T, A)$ because $\nu(\lambda[A], x) = \nu(T, A)$ for all $x \in A \setminus A'$. □

Theorem 4.7.3 (Siu's formula) *For any current $T \in \mathcal{D}_p^+(\Omega)$, there is a unique decomposition*

$$T = \sum_j \lambda_j [A_j] + R, \tag{4.7.4}$$

where $\lambda_j > 0$, $[A_j]$ are the integration currents over irreducible $p-$dimensional varieties A_j, and $R \in \mathcal{D}_p^+(\Omega)$ is such that $\dim E_c(R) < p$ for every $c > 0$. The values λ_j are the generic Lelong numbers of T along the varieties A_j.

Proof. Let $\{A_j\}$ be $p-$dimensional irreducible components of the set

$$\{E_c(T)\}_{c \in \mathbb{Q}_+},$$

and let $\lambda_j = \nu(T, A_j)$. Then the sequence of currents

$$R_1 := T - \lambda_1[A_1] \in \mathcal{D}_p^+(\Omega),\ R_2 := R_1 - \lambda_2[A_2] \in \mathcal{D}_p^+(\Omega), \ldots$$

decreases to a current $R \in \mathcal{D}_p^+(\Omega)$, which gives us (4.7.4). Furthermore, $\dim E_c(R) < p$ for every $c > 0$ because $\nu(R, A_j) = 0$ for all j. □

Some components of lower dimension can actually occur in $E_c(R)$, however $\chi_A R = 0$ for any $p-$dimensional variety A.

4.8 King–Demailly formula

Siu's structural formula can be specified for the case when $T = (dd^c \log|f|)^p$ for a holomorphic mapping $f = (f_1, \ldots, f_q) : \Omega \to \mathbb{C}^q$. Let $\{A_j\}$ be the irreducible components of its zero set $A_f = f^{-1}(0)$. Consider $u = \log|f|$. If $\operatorname{codim} A_f = \min_j \operatorname{codim} A_j \geq l$, then $(dd^c u)^l$ is well defined.

When dealing with holomorphic mappings, it is convenient to consider the corresponding *holomorphic chains*, i.e., the currents

$$Z_f = \sum_j m_j [A_j]$$

where the summation runs over all $p-$codimensional components A_j of the variety A_f, and $m_j \in \mathbb{Z}_+$ are the generic multiplicities of f at A_j (i.e., the generic Lelong numbers of $(dd^c \log|f|)^p$ along A_j; we will explain soon why these are the same).

The following result (originally established by King for $q = p \le n$, see [GK73], and then extended by Demailly [D87], [Dbook] to the general case) represents holomorphic chains as singular parts of Monge-Ampère currents.

Theorem 4.8.1 (King–Demailly formula) *Let $f : \Omega \to \mathbb{C}^q$ be a holomorphic mapping,* $\operatorname{codim} A_f = p$. *Then the currents $(dd^c \log|f|)^l$ and $\log|f|(dd^c \log|f|)^l$ with $l < p$ have locally integrable coefficients, and*

$$(dd^c \log|f|)^p = Z_f + R, \tag{4.8.5}$$

where Z_f is the corresponding holomorphic chain and $R \in \mathcal{D}^+_{n-p}(\Omega)$ is such that $\chi_{A_f} R = 0$ and the sets $E_c(R)$, $c > 0$, are of codimension at least $p + 1$.

Proof. For $l < p$, the currents $(dd^c \log|f|)^l$ and $\log|f|(dd^c \log|f|)^l$ are well defined on Ω and have smooth coefficients on $\Omega \setminus A_f$; since $\dim A_f < n - l$, they cannot charge A_f. By Siu's formula (Theorem 4.7.3),

$$(dd^c \log|f|)^p = \sum_j \lambda_j [A_j] + R$$

with $\lambda_j > 0$ generic Lelong numbers of $(dd^c \log|f|)^p$ along A_j. We will see soon that they are precisely the multiplicities m_j. □

Examples 4.8.2 Particular cases:

1) $p = q = 1$: no condition on A is required, $R = 0$, so we have the *Lelong-Poincarè equation*

$$dd^c \log|f| = Z_f.$$

2) $p = q \le n$: $R = 0$ because in this case the restriction of $\log|f|$ to $p-$dimensional complex planes L are maximal psh function on $L \setminus A_f$ and so, $(dd^c \log|f|)^p = 0$ outside A_f. This gives *King's formula* [GK73]

$$(dd^c \log|f|)^q = \sum_j m_j [A_j] = Z_f.$$

4.9 Lelong numbers and multiplicities

We have seen that the Lelong number of $dd^c \log|f|$ for a holomorphic function $f : \Omega \to \mathbb{C}$ at $x \in \Omega$ equals the multiplicity of the zero of f at x, and in that case it is just the vanishing order of f at x (the least degree of its monomials about x). For holomorphic mappings to $\mathbb{C}^q$, $q > 1$, the multiplicities are defined in a different way.

In the simplest situation $q = p = n$ (p being the codimension of the zero set at x), f is a finite covering of a neighborhood U_0 of 0 by a neighborhood V_x of x, and the multiplicity of f at x is just the number of the sheets the multiplicity of an equidimensional mapping f . In other words, it is the generic number of solutions $y \in U_x$ to the equation $f(y) = a$, $a \in \mathbb{B}_\epsilon$.

When $q > n$ and still the codimension p equals n, the multiplicity of f is the one for the mapping $f' : \Omega \to \mathbb{C}^n$ whose components are generic linear combinations of the components of f.

When $p < n$, one considers restrictions of the mapping f to generic $q-$dimensional complex planes passing through x.

Theorem 4.9.1 *If the codimension of the zero set A_f of f at x equals p, then $\nu((dd^c \log|f|)^p, x)$ equals the multiplicity of f at x.*

Proof. This is easy to check if $f : \mathbb{C}^n_0 \to \mathbb{C}^n_0$ and $p = n$, i.e., x is an isolated point of $f^{-1}(0)$. Let s be the number of the sheets of the covering. Then

$$\nu((dd^c \log|f|)^n, x) = \nu([V_x], \log|f|) = \nu(f_*[V_x], 0) = s\,\nu([U_0], 0) = s.$$

A reduction of the general situation to this case is a bit technical; we omit the details. □

Note also that A_f is the intersection of the zero sets A_{f_k} of the components f_k of f. A corresponding formula for the case $p = q \le n$ represents the holomorphic chain Z_f as the intersection product of the divisors of the components f_k of the mapping f. Denote $u_k = \log|f_k|$.

Theorem 4.9.2 *If the zero sets A_{f_k} satisfy condition*

$$\dim A_{j_1} \cap \ldots \cap A_{j_m} \le n - m$$

then for all m-tuples $(j_1, \ldots, j_m)$ and all $m \le q$, then

$$dd^c u_1 \wedge \ldots \wedge dd^c u_q = Z_f.$$

Proof. By the Lelong-Poincarè equation, $dd^c u_k = Z_{f_k}$, so

$$dd^c u_1 \wedge \ldots \wedge dd^c u_q = Z_{f_1} \wedge \ldots \wedge Z_{f_q}.$$

Being supported by A_f, it has the form $\sum_j \alpha_j [A_j]$, where $\{A_j\}$ are the irreducible components of A_f and $\alpha_j > 0$ are the intersection multiplicities of $Z_{f_1}, \ldots, Z_{f_q}$ along A_j (proof by induction). As is known in intersection theory, these are precisely the multiplicities of the mapping f. □

5 Evaluation of residual Monge-Ampère masses

Computing Lelong numbers of psh functions is a relatively easy task, while computation of the Lelong numbers for the Monge-Ampère currents $(dd^c u)^m$, $m > 1$, is much more complicated. Even for $u = \log |f|$ for holomorphic mappings f (i.e, for computing multiplicities of the mappings), no general explicit formulas are available, and we have to restrict ourselves to certain *bounds* that might become equalities in 'generic' situations.

No upper bound of $\nu((dd^c u)^m, x)$ in terms of $\nu(u, x)$ is possible. Nevertheless, it seems to be unknown if there exists a psh function with zero Lelong number and nonzero residual Monge-Ampère mass.

As to estimates from below, a standard bound (following from Demailly's comparison theorem) is

$$\nu((dd^c u)^m, x) \geq [\nu(u, x)]^m,$$

which is far from being an equality unless $u(z)$ is essentially $c \log |z - x|$. More precise relations can be obtained by means of more refined characteristics of local behaviour of a function, e.g., directional Lelong numbers. This way we get, in particular, relations for the multiplicities of the mappings in terms of their *Newton polyhedra*.

This part is based on [LeR99] and [R00]. For a nice presentation of the real Monge-Ampère operator, see [RaT77]. Kushnirenko-Bernstein's theorems are presented in [AYu79], [Ku76].

5.1 Reduction to local indicators

We will set here $x = 0$ and consider the classical Lelong numbers of the Monge-Ampère currents $(dd^c u)^m$ and the mixed Monge-Ampère currents

$dd^c u_1 \wedge \ldots \wedge dd^c u_q$ at 0. A basic tool will be again Demailly's comparison theorem, which implies the following: *assuming the Monge-Ampère currents* $dd^c u_1 \wedge \ldots \wedge dd^c u_q$ *and* $dd^c v_1 \wedge \ldots \wedge dd^c v_q$ *well defined,* $2 \le q \le n$*, let*

$$\limsup_{z \to 0} \frac{u_k(z)}{v_k(z)} = l_k < \infty, \ 1 \le k \le q;$$

then

$$\nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q, 0) \le l_1 \ldots l_q \, \nu(dd^c v_1 \wedge \ldots \wedge dd^c v_q, 0). \tag{5.1.1}$$

Since $u_k(z) \le \nu(u_k, 0) \log |z| + O(1)$, this gives us

$$\nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q, 0) \le \nu(u_1, 0) \ldots \nu(u_q, 0). \tag{5.1.2}$$

For a more precise bound, choose $v_k = \Psi_{u_k}$, the indicators of u_k at 0. Recall that the indicator Ψ_u of $u \in \mathrm{PSH}_0$ is a toric psh function in the unit polydisk, defined as

$$\Psi_u(z) := \psi_u(\log |z_1|, \ldots, \log |z_n|),$$

where $\psi_u(s) = -\nu(u, 0, -s)$, $s \in \mathbb{R}^n_-$, and $\nu(u, 0, a)$ are directional Lelong numbers.

Since $u \le \Psi_u + C$ near the origin, (5.1.1) implies

Theorem 5.1.1 *If* $dd^c u_1 \wedge \ldots \wedge dd^c u_q$ *is well defined near the origin, then*

$$\nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q, 0) \ge \nu(dd^c \Psi_{u_1} \wedge \ldots \wedge dd^c \Psi_{u_q}, 0).$$

For $u \in L^\infty_{loc}(\Omega \setminus \{0\})$, the operator $(dd^c u)^n$ is well defined, and the value

$$\mathcal{R}_u := \nu((dd^c u)^n, 0)$$

is the *residual measure* of $(dd^c u)^n$ at 0. In this situation, Ψ_u is a maximal psh function on $\mathbb{D}^n \setminus \{0\}$ and so, by Theorem 2.3.1, $(dd^c \Psi_u)^n = 0$ on $D \setminus \{0\}$. Therefore,

$$(dd^c \Psi_u)^n = N_u \, \delta_0$$

with

$$N_u = \mathcal{R}_{\Psi_u},$$

the Newton number of u at 0 (the reason for using the name will be clarified soon).

Corollary 5.1.2 *If $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}(\Omega \setminus \{0\})$, then $\mathcal{R}_u \geq N_u$.*

More generally, let $\{u\}$ be an $n-$tuple of psh functions $u_1, \dots, u_n$ in Ω. If the Monge-Ampère current $dd^c u_1 \wedge \ldots \wedge dd^c u_n$ is well defined near 0, then its residual Monge-Ampère mass

$$\mathcal{R}_{\{u\}} := (dd^c u_1 \wedge \ldots \wedge dd^c u_n)(0)$$

has the bound

$$\mathcal{R}_{\{u\}} \geq N_{\{u\}},$$

where

$$N_{\{u\}} = \mathcal{R}_{\{\Psi_u\}} = (dd^c \Psi_{u_1} \wedge \ldots \wedge dd^c \Psi_{u_n})(0).$$

To make all this reasonable, one has to look for good bounds for the Newton numbers.

5.2 Geometric interpretation: volumes

More sharp bounds can be obtained by precise calculation of the Monge-Ampère masses of the indicators. This can be done by switching to the *real* Monge-Ampère operator.

Definition 5.2.1 Let U be a *toric* psh function in the unit polydisk $\mathbb{D}^n$, i.e., $U(z) = U(|z_1|, \dots, |z_n|) \in PSH(\mathbb{D}^n)$. Then the function

$$h(t) := U(\exp(t_1), \dots, \exp(t_n)), \quad t \in \mathbb{R}^n_-,$$

is convex in t and increasing in each t_k. We call it *the convex image* of U.

Assume, in addition, $u \in L^\infty(\mathbb{D}^n)$. By elementary computations,

$$(dd^c U)^n = n!(2\pi)^{-n} \mathcal{MA}_\mathbb{R}[h]\, d\theta, \quad z_k = \exp\{t_j + i\theta_j\},$$

where $\mathcal{MA}_\mathbb{R}$ is the *real Monge-Ampère operator*[4]. For h smooth,

$$\mathcal{MA}_\mathbb{R}[h] = \det \left(\frac{\partial^2 h}{\partial t_j \partial t_k} \right) dt,$$

and it extends, as a measure-valued operator, to all convex functions h. Furthermore,

$$\mathcal{MA}_\mathbb{R}[h](F) = \mathrm{Vol}(G_h(F)) \tag{5.2.3}$$

[4]For a thorough treatment of the real Monge-Ampère operator, see [RaT77].

for Borel sets $F \subset\subset \mathbb{R}^n_-$, where

$$G_h(F) = \bigcup_{t^0 \in F} \{a \in \mathbb{R}^n : h(t) \geq h(t^0) + \langle a, t - t^0 \rangle \ \forall t \in \mathbb{R}^n_-\}$$

is the *gradient image* of F for the surface $\xi = h(t)$.

The terminology and the proof come from the smooth situation where $\mathcal{MA}_{\mathbb{R}}[h]$ equals the Jacobian determinant $J_{\nabla h}$ for the gradient mapping ∇h, $G_h(F) = \nabla h(F)$, and the equation (5.2.3) is just the coordinate change formula.

So, for any Borel, n-circled (toric, Reinhardt) set $E \subset\subset \mathbb{D}^n$ and its *logarithmic image*

$$\log E = \{t \in \mathbb{R}^n_- : (\exp t_1, \ldots, \exp t_n) \in E\},$$

we have

$$(dd^c U)^n(E) = n! \operatorname{Vol} G_h(\log E).$$

Definition 5.2.2 $\Psi \in \mathrm{PSH}_-(\mathbb{D}^n)$ is an *(abstract) indicator* if Ψ is a toric function whose convex image $\psi(t) := \Psi(\exp(t_1), \ldots, \exp(t_n))$ is positive homogeneous: $\psi(ct) = c\psi(t)$ for all $c > 0$.

In other words, ψ is the restriction to $\mathbb{R}^n_-$ of the support function of a convex subset of $\mathbb{R}^n_+$:

$$\psi(t) = \sup\{\langle a, t \rangle : a \in \Gamma_\Psi\}, \ t \in \mathbb{R}^n_-,$$

where

$$\Gamma_\Psi = \{a \in \mathbb{R}^n_+ : \langle a, t \rangle \leq \psi(t) \ \forall t \in \mathbb{R}^n_-\}. \tag{5.2.4}$$

Denote $U = \max\{\Psi, -1\}$ with Ψ an indicator. Then the current $(dd^c U)^n$ is supported by the set $E_\Psi = \{z : \Psi(z) = -1\}$, and

$$G_h(\log E_\Psi) = \mathbb{R}^n_+ \setminus \Gamma_\Psi.$$

Furthermore, if $\Psi \in L^\infty_{loc}(\mathbb{D}^n \setminus \{0\})$, then

$$(dd^c U)^n(\mathbb{D}^n) = (dd^c \Psi)^n(\mathbb{D}^n) = \mathcal{R}_\Psi.$$

For any convex and *complete* subset Γ of $\mathbb{R}^n_+$ (i.e., $\Gamma + \mathbb{R}^n_+ \subset \Gamma$), we put

$$\operatorname{Covol}(\Gamma) = \operatorname{Vol}(\mathbb{R}^n_+ \setminus \Gamma),$$

the *covolume* of Γ. So we get

Proposition 5.2.3 *The MA-mass of an indicator* $\Psi \in L^\infty_{loc}(\mathbb{D}^n \setminus \{0\})$ *is*

$$\mathcal{R}_\Psi = n!\,\mathrm{Covol}(\Gamma_\Psi),$$

where Γ_Ψ *is defined in (5.2.4).*

When $\Psi = \Psi_u$, the set

$$\Gamma_\Psi = \Gamma_u := \{a \in \mathbb{R}^n_+ : \nu(u, 0, a) \le \langle a, b\rangle \ \forall b \in \mathbb{R}^n_+\}. \qquad (5.2.5)$$

Thus Corollary 5.1.2 gives us

Theorem 5.2.4 *If* u *has isolated singularity at* 0*, then*

$$\mathcal{R}_u \ge N_u = n!\,\mathrm{Covol}(\Gamma_u). \qquad (5.2.6)$$

To compute the mass of the corresponding mixed Monge-Ampère operators of indicators, we consider a (unique) form $\mathrm{Covol}\,(\Gamma_1, \dots, \Gamma_n)$ on n-tuples of complete convex subsets $\Gamma_1, \dots, \Gamma_n$ of $\mathbb{R}^n_+$ which is multilinear with respect to Minkowsky's addition and such that for every convex complete Γ with finite covolume we have $\mathrm{Covol}\,(\Gamma, \dots, \Gamma) = \mathrm{Covol}\,\Gamma$. The form can be shown to be well defined on all n-tuples $\Gamma_1, \dots, \Gamma_n$ such that $\cup_j \Gamma_j$ has finite covolume.

Theorem 5.2.5 *Let* $\{u\} = u_1, \dots, u_n$*. If* $dd^c u_1 \wedge \ldots \wedge u_n$ *is well defined, then*

$$\mathcal{R}_{\{u\}} \ge n!\,\mathrm{Covol}(\Gamma_{u_1}, \dots, \Gamma_{u_n}).$$

Note that if $u = \log|z|$, then $\Gamma_u = \Delta^c := \{a \in \mathbb{R}^n_+ : \sum_k a_k \ge 1\}$, the complement to the standard simplex. Therefore, for $\{u\} = u_1, \dots, u_q$, $1 \le q < n$, with the Monge-Ampère current $dd^c u_1 \wedge \ldots \wedge dd^c u_q$ well defined, we have the bound

$$\mathcal{R}_{\{u\}} \ge n!\,\mathrm{Covol}(\Gamma_{u_1}, \dots, \Gamma_{u_q}, \Delta^c, \dots, \Delta^c).$$

5.3 Applications to holomorphic mappings: Newton polyhedra

For a function $u = \log|f|$ generated by a holomorphic mapping $f : \mathbb{C}_0^n \to \mathbb{C}_0^q$ $(q \geq n)$ with isolated zero, the residual Monge-Ampère mass $\mathcal{R}_u$ of $(dd^c u)^n$ at 0 is the multiplicity m_f of f at the origin.

When $q = n$ and the zero sets of the components of f are properly intersected, inequality (5.1.2) gives us the bound

$$m_f \geq m_{f_1} \dots m_{f_n}$$

via the multiplicities of the components of the mapping f, which is *a local variant of Bezout's theorem*.

As we saw in (1.6.12), the indicator of $u = \log|f|$ for a holomorphic mapping f computes as

$$\Psi_u(z) = \sup\left\{\log|z^J| : J \in \omega_0\right\},$$

where $\omega_0 \subset \mathbb{Z}_+^n$ is the collection of multi-indices J such that z^J has nonzero coefficient in the Taylor expansion of at least one component of f. Therefore, its convex image ψ_u has the representation

$$\psi_u(t) = \sup\{\langle t, J\rangle : \ J \in \omega_0\}.$$

This means that the set Γ_u (5.2.5) is the convex hull Γ_f of the set ω_0, and Γ_f is known as the *Newton polyhedron* for the mapping f at 0. Therefore, a particular case of Theorem 5.2.4 recovers the bound

$$m_f \geq n!\,\mathrm{Covol}(\Gamma_f)$$

obtained for $q = n$ by Kushhnirenko (1975) by means of analytic and algebraic techniques. The corresponding specification of Theorem 5.2.5 gives a Kushnirenko-Bernstein's type result.

Note that for holomorphic mappings f to $\mathbb{C}^q$, $q < n$, with the zero set of codimension q, Theorem 5.2.5 gives the bound

$$m_f \geq n!\,\mathrm{Covol}(\Gamma_{u_1}, \dots, \Gamma_{u_q}, \Delta^c, \dots, \Delta^c),$$

where $u_j = \log|f_j|$, the sets Γ_{u_j} are the Newton polyhedra of the functions f_j at 0, and $\Delta^c = \{a \in \mathbb{R}_+^n : \sum_k a_k \geq 1\}$.

So, methods of pluripotential theory are quite powerful to produce, in a simple and unified way, efficient bounds for multiplicities of holomorphic mappings.

6 Open questions

Here we list just a few problems on psh singularities.

If $u \in \mathrm{PSH}_x$ has isolated singularity, then $(dd^c u)^n(x) \geq [\nu(u,x)]^n$, while no reverse bound is possible. Indeed, take $u = \max\{k \log|z_1|, \log|z_2|\}$, $k > 0$, then $\nu(u,0) = 1$ while $(dd^c u)^n(0) = k$.

Question 1 (*Zero Lelong Number Problem*; V. Guedj, A. Rashkovskii, 1999): *Is the implication*

$$(P1) \qquad \nu(u,x) = 0 \;\Rightarrow\; (dd^c u)^n(x) = 0$$

true whenever $(dd^c u)^n$ is well defined (for example, if u is locally bounded outside x)? (This is Question 7 from [DiGuZe16].)

This is true (by Demailly's comparison theorem) if u has the lower bound

$$u(x) \geq c \log|z - x| + O(1) \tag{6.0.1}$$

for some $c > 0$. By the Łojasiewicz inequality, any holomorphic mapping f with isolated zero at x satisfies

$$|f(z)| \geq |z - x|^\gamma$$

for some $\gamma > 0$, so psh functions with analytic singularities satisfy (6.0.1).

It can be shown that if u is locally bounded outside x, there exists a function v which is locally bounded and maximal outside x and such that $\nu(v,x) = \nu(u,x)$, $(dd^c v)^n(x) = (dd^c u)^n(x)$.

Question 2: Does there exist $v \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}(\Omega \setminus \{x\})$, maximal in $\Omega \setminus \{x\}$ and such that

$$\limsup_{z \to x} \frac{u(z)}{\log|z - x|} = \infty?$$

Question 1 may be approached by approximating u by functions u_m with analytic singularities (Demailly's approximation, Theorem 4.3.2) for which (P1) is true. It is known that $\nu(u_m, x) \to \nu(u,x)$, however it is not clear if their residual Monge-Ampère at x converge to that of u.

Question 3 (Demailly): *Is it true that $(dd^c u_m)^n(x) \to (dd^c u)^n(x)$?*

More information on these questions can be found in [R16]. Other open problems on psh singularities and related topics (including *quasi-psh functions*) are presented in [DiGuZe16].

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